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The energy theorem in the impact of a string vibrating against a pointshaped obstacle

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Analisi matematica. — *The energy theorem in the impact of a string vibrating against a pointshaped obstacle.* Nota di CLAUDIO CITRINI (*), presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si ricava l'uguaglianza dell'energia per una corda vibrante contro un ostacolo puntiforme, in un caso studiato da Amerio. Non si hanno perdite di energia, e, se i vincoli sono fissi, vale il principio di conservazione dell'energia.

1. The aim of this paper is to obtain the energy relations for the vibrating string equation, subject to an unilateral constraint, in a case studied by Amerio [1]. For an analogous case, with a different kind of constraint, see also Amerio and Prouse [2], Citrini [3], [4]. The problem solved in [1] is as follows: given in the (x, t) plane a strip

$$Z = \{(x, t) : t \geq 0, \quad p(t) \leq x \leq q(t)\} \tag{1.1}$$

and a line $\Lambda : \{x = \lambda(t)\} \subset Z$, we look for a function $y = y(x, t) \in C^0(Z)$, satisfying the following conditions:

$$y_{\xi\eta} = f \quad \text{in} \quad Z - \Lambda \tag{1.2}$$

$$y(x, 0) = \varphi(x) \quad , \quad y_t(x, 0) = \psi(x) \tag{1.3}$$

$$y(p(t), t) = A(t) \quad , \quad y(q(t), t) = B(t) \tag{1.4}$$

$$y(\lambda(t), t) \geq \beta(t) \tag{1.5}$$

where f, φ, ψ, A, B and β are given, and

$$\xi = (x + t)/\sqrt{2} \quad , \quad \eta = (-x + t)/\sqrt{2} \tag{1.6}$$

are the characteristic coordinates for (1.2) Let us observe that, in the sense of distributions, (1.2) is equivalent to the classical equation

$$y_{tt} - y_{xx} = 2f.$$

Under suitable hypotheses, in [1], existence and uniqueness of the solution y are proved, and an explicit expression is given. We shall replace the hypo-

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theses of [1] by the following, a little more restrictive:

$$(1.7)\qquad \begin{cases} p(t), q(t) \text{ are Lipschitz continuous on } [0, +\infty), \\ p(t) < q(t), \quad |p'(t)|, |q'(t)| \leq k < 1, \end{cases}$$

$$(1.8)\qquad \begin{cases} \lambda(t) \quad \text{is Lipschitz continuous on } [0, +\infty), \\ p(t) < \lambda(t) < q(t), \quad |\lambda'(t)| \leq k < 1, \\ \lambda'(t) \text{ vanishes at most in countably many points and intervals} \\ \text{in } [0, T], \forall T > 0; \end{cases}$$

$$(1.9)\qquad f \in L^2(Z_T), \quad \text{where } Z_T = Z \cap \{t \leq T\}, \forall T > 0;$$

$$(1.10)\qquad A, B \in H^1([0, T]) \quad \forall T > 0;$$

$$(1.11)\qquad \varphi', \psi \in L^2([p(0), q(0)]);$$

$$(1.12)\qquad \beta(0) \leq \varphi(\lambda(0)), \quad \beta \in H^1([0, T]) \quad \forall T > 0.$$

By (1.7), (1.9), (1.10) and (1.11), the solution $u(x, t)$ of the free problem (1.2), (1.3) and (1.4) belongs to the class W^* of the functions $\in C^0(Z)$, with $w_{\xi\eta} \in L^2(Z_T)$ $\forall T > 0$ and whose partial derivatives w_ξ are square-integrable on every segment of characteristic ξ contained in Z (and likewise for w_η). Moreover, the trace $u(\lambda(t), t)$ of u on Λ belongs to $H^1([0, T])\,\forall T > 0$: for this function, in fact, Vitali condition is satisfied; hence, it is absolutely continuous.

Let us observe that (1.8) implies that the longitudinal speed of the point-shaped obstacle is strictly less than the propagation speed along the string.

2. We first prove some results about the Π problem (cfr. Amerio [1]). Let

$$(2.1)\qquad \Lambda_R : \{\xi = \xi(t), \eta = \eta(t), 0 \leq t \leq c\}$$

be a line contained in the rectangle $R = \{0 \leq \xi \leq a, 0 \leq \eta \leq b\}$, with $\xi(t)$, $\eta(t)$ strictly increasing functions of t, satisfying

$$(2.2)\qquad \xi(0) = \eta(0) = 0 \quad , \quad \xi(c) = a \quad , \quad \eta(c) = b.$$

We look for a function $\Gamma(P) \in C^0(R)$, subject to the conditions

$$(2.3)\qquad \begin{cases} \text{(i)} & \Gamma(\xi, 0) = 0, \quad \Gamma(0, \eta) = 0 \\ \text{(ii)} & \Gamma(\xi(t), \eta(t)) \geq \omega(t) \\ \text{(iii)} & \Gamma_{\xi\eta} \geq 0 \\ \text{(iv)} & \operatorname{supp} \Gamma_{\xi\eta} \subset \{P \in \Lambda_R : \Gamma(P(t)) = \omega(t)\} \end{cases}$$

(II problem), where $\omega(t)$ is a given function $\in C^0([0, c])$.

The following lemmas then hold.

LEMMA 1. *Under the hypotheses*

$$\begin{cases} \xi(t), \eta(t) & \text{Lipschitz continuous on } [0, c], \\ 0 < k' \leq \xi', \eta' \leq k''; \end{cases} \tag{2.4}$$

$$\omega \in H^1([0, c]) \tag{2.5}$$

the (unique) solution Γ *of the problem* (2.3) $\in H^1(R)$, *and its derivatives* $\Gamma_\xi \in L^2([0, a]) \, \forall \eta \in [0, b]$, *and* $\Gamma_\eta \in L^2([0, b]) \, \forall \xi \in [0, a]$.

Proof. Let $t = t_1(\xi)$ and $t = t_2(\eta)$ be the inverse functions of (2.1). By (2.4) it is, a.e.,

$$0 < 1/k'' \leq t_1', t_2' \leq 1/k'. \tag{2.6}$$

Setting then in $[0, c]$:

$$\Omega(t) = \max_{0 \leq \tau \leq t} \omega(\tau)^+ \tag{2.7}$$

the (unique) solution of (2.3) is given, as proved in [1], by the function

$$\Gamma(\xi, \eta) = \begin{cases} \Omega(t_1(\xi)) & \text{for } \eta \geq \eta(t_1(\xi)) \\ \Omega(t_2(\eta)) & \text{for } \xi \geq \xi(t_2(\eta)). \end{cases} \tag{2.8}$$

By (2.5), (2.7), $\Omega(t)$ is absolutely continuous, and its derivative satisfies the inequality $0 \leq \Omega'(t) \leq \omega'(t)^+$: therefore it is also:

$$\Omega \in H^1([0, c]). \tag{2.9}$$

Now, by (2.8), it is

$$\Gamma_\xi = \begin{cases} \Omega'(t_1(\xi)) \, t_1'(\xi) & \text{for } \eta > \eta(t_1(\xi)) \\ 0 & \text{for } \xi > \xi(t_2(\eta)) \end{cases} \tag{2.10}$$

and then by (2.6) (2.9) the result follows. The argument for Γ_η is similar.

Consider now the solution y of (1.2) in R, given by $y = u + \Gamma$, with $\Lambda_R = \Lambda \cap R$, Γ solution of (2.3) and

$$y(\xi(t), \eta(t)) \geq \beta(t) \quad \text{in } [0, c]. \tag{2.11}$$

Note that (2.4) follow from (1.8), because, by (1.6), it is:

$$\xi(t) = (t + \lambda(t))/\sqrt{2}, \; \eta(t) = (t - \lambda(t))/\sqrt{2}. \tag{2.12}$$

Being $\omega(t) = \beta(t) - u(\xi(t), \eta(t))$, by the properties of u and by (1.12), (2.5) also is verified, and Lemma 1 holds. We are now able to prove the following

LEMMA 2. *The relation*

$$(2.13) \qquad \frac{1}{\sqrt{2}} \int_{\partial R} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = 2 \int_{\mathring{R}} y_t f \, dR + \int_{\Lambda_R} (y_t^+ + y_t^-) \, \Omega'(t) \, dt$$

(energy equality for Π *problem) holds.*

All the terms in (2.13) are meaningful by Lemma 1, (1.8) (1.9) (2.9) and (2.10). The boundary ∂R is travelled *clockwise*.

For the proof, let us consider the curved triangles:

$$R' = \{0 \leq \xi \leq \xi(t_2(\eta)), 0 \leq \eta \leq b\}$$

$$R'' = \{0 \leq \eta \leq \eta(t_1(\xi)), 0 \leq \xi \leq a\}$$

into which R is divided by Λ_R. If O(0, 0), A(a, 0), B(0, b) and C(a, b) are the vertices of R, it is:

$$(2.14) \qquad \begin{aligned} \partial R' &= OB \cup BC \cup (-\Lambda_R) \\ \partial R'' &= \Lambda_R \cup CA \cup AO. \end{aligned}$$

We can now apply the Stokes theorem in R' and R'', because in $\mathring{R}'$ and $\mathring{R}''$ it is $y_{\xi\eta} = f$, taking care of replacing the derivatives y_ξ and y_η, which do not exist on Λ_R, by right or left derivatives, which exist a.e. Being

$$(2.15) \qquad y_t = (y_\xi + y_\eta)/\sqrt{2}$$

we obtain

$$\frac{1}{\sqrt{2}} \int_{\partial R'} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = \frac{1}{\sqrt{2}} \int_{\mathring{R}'} (2\, y_\xi \, y_{\xi\eta} + 2\, y_\eta \, y_{\xi\eta}) \, dR = 2 \int_{\mathring{R}'} y_t f \, dR,$$

and likewise

$$\frac{1}{\sqrt{2}} \int_{\partial R''} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = 2 \int_{\mathring{R}''} y_t \cdot f \, dR,$$

from which, by addition,

$$\frac{1}{\sqrt{2}} \int_{\partial R' \cup \partial R''} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = 2 \int_{\mathring{R}} y_t f \, dR.$$

We obtain thus, by (2.14):

$$(2.16)\qquad \frac{1}{\sqrt{2}}\int_{\partial R} y_\xi^2\, d\xi - y_\eta^2\, d\eta - 2\int_{\mathring{R}} y_t f\, dR = -\frac{1}{\sqrt{2}}\int_{\Lambda'_R\cup\Lambda''_R} y_\xi^2\, d\xi - y_\eta^2\, d\eta\,.$$

In the right member of (2.16), the line Λ'_R (resp. Λ''_R) coincides with Λ_R, which we regard as belonging to the boundary of R′ (resp. R″) and as travelled in the sense of decreasing (resp. increasing) t. The integrals then are different, and we must set, by (2.10):

$$y_\xi = \begin{cases} u_\xi + \Gamma_\xi^- = u_\xi + \Omega'\, t'_1 & \text{on } \Lambda'_R \\ u_\xi + \Gamma_\xi^+ = u_\xi & \text{on } \Lambda''_R \end{cases}$$

and likewise

$$y_\eta = \begin{cases} u_\eta + \Gamma_\eta^+ = u_\eta & \text{on } \Lambda'_R \\ u_\eta + \Gamma_\eta^- = u_\eta + \Omega'\, t'_2 & \text{on } \Lambda''_R\,. \end{cases}$$

We obtain then, taking into account that $t'_1\,\xi' = t'_2\,\eta' = 1$:

$$-\frac{1}{\sqrt{2}}\int_{\Lambda'_R\cup\Lambda''_R} y_\xi^2\, d\xi - y_\eta^2\, d\eta =$$

$$= \frac{1}{\sqrt{2}}\int_{\Lambda_R} \{[(u_\xi + \Omega'\, t'_1)^2 - u_\xi^2]\, d\xi - [u_\eta^2 - (u_\eta + \Omega'\, t'_2)^2]\, d\eta\} =$$

$$= \frac{1}{\sqrt{2}}\int_{\Lambda_R} 2\,(u_\xi + u_\eta)\,\Omega'\, dt + \frac{1}{\sqrt{2}}\int_{\Lambda_R} (\Omega'\, t'_1 + \Omega'\, t'_2)\,\Omega'\, dt =$$

$$= 2\int_{\Lambda_R} u_t\,\Omega'\, dt + \frac{1}{\sqrt{2}}\int_{\Lambda_R} (\Gamma_\xi^- + \Gamma_\eta^-)\,\Omega'\, dt.$$

Let us now consider a point of Λ_R, where $\lambda' > 0$. It is then

$$\Gamma_\eta^- = \sqrt{2}\,\Gamma_t^- \quad , \quad \Gamma_\xi^- = \sqrt{2}\,\Gamma_t^+$$

from which

$$(2.17)\qquad \frac{1}{\sqrt{2}}(\Gamma_\xi^- + \Gamma_\eta^-) = \Gamma_t^- + \Gamma_t^+\,.$$

The same relation is obtained if $\lambda' < 0$. If, on the contrary, $\lambda' \equiv 0$ in an interval, we obtain

$$\frac{1}{\sqrt{2}} D_t^-\,\Gamma(\xi(t), \eta(t)) = \Gamma_t^- = \Omega'(t^-)$$

but $\Omega(t)$ has a derivative a.e., and thereafter $\Gamma_t^- = \Gamma_t^+$ a.e.. By (1.8), (2.17) holds a.e. on Λ_R, and (2.13) is proved.

Observation. (2.13) continues to hold if we replace (2.3 i) by non homogenous conditions, as $\Gamma(\xi, 0) = \Phi(\xi)$, $\Gamma(0, \eta) = \Psi(\eta)$, $\Phi \in H^1([0, a])$, $\Psi \in H^1([0, b])$, $\Phi(0) = \Psi(0)$. Setting, in fact, $\bar{u} = u + \Phi(\xi) + \Psi(\eta) - \Phi(0)$, then $\bar{u}$ enjoys the same properties as u. Furthermore we have $y = u + \Gamma = \bar{u} + \bar{\Gamma}$, and $\bar{\Gamma}$ satisfies a homogeneous problem (cfr. [1], § 2, Observation). *We shall write everywhere*, for the sake of simplicity, $\Omega(t)$ *instead of* $\bar{\Omega}(t)$.

3. Proceeding now to the construction of the solution y on the whole of Z, we note that Γ enjoys, in $Z - \Lambda$, the same properties as u. Hence, if A is an arbitrary open set with boundary ∂A Lipschitz continuous, such that $\bar{A} \subset Z - \Lambda$, we have:

$$\frac{1}{\sqrt{2}} \int_{\partial A} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = 2 \int_A y_t f \, dA. \tag{3.1}$$

(3.1) continues to hold in all the domains $T_1 T_2 S_1 S_2 \cdots$ (cfr. fig. 5 of [1]) into which Z is decomposed in a well known way, in order to obtain $y(x, t)$ as solution of elementary problems. For every Π problem which must be solved, however, (2.13) holds, by the above observation. We have therefore, for every domain Δ with a Lipschitz continuous boundary $\partial\Delta$:

$$\frac{1}{\sqrt{2}} \int_{\partial\Delta} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = 2 \int_\Delta y_t f \, d + \int_{\Lambda\cap\Delta} (y_t^+ + y_t^-) \, \Omega'(t) \, dt. \tag{3.2}$$

If, in particular, we choose $\Delta = Z \cap \{T_1 \le t \le T_2\}$, we have $\partial\Delta = \Xi_2 \cup (-\Xi_1) \cup \Xi_p \cup (-\Xi_q)$, where

$$\Xi_i = \{t = T_i \,,\, p(T_i) \le x \le q(T_i)\} \qquad i = 1, 2$$

$$\Xi_p \text{ (resp. } \Xi_q) = \{x = p(t) \text{ (resp. } q(t)) \,,\, T_1 \le t \le T_2\}.$$

Since the vibrating string energy (at time T_i) is given (cfr. [4]) by

$$E(T) = \frac{1}{\sqrt{2}} \int_{\Xi i} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta \qquad i = 1, 2$$

and

$$L_p = \frac{1}{\sqrt{2}} \int_{\Xi p} y_\xi^2 \, d\xi - y_\eta^2 \, d\eta = \int_{\Xi p} [\tfrac{1}{2}(y_x^2 + y_t^2) \, dx + y_x \, y_t \, dt]$$

represents the change in energy due to the variation of the string length, and to the work of the constraint $x = p(t)$ (and likewise for L_q), (3.2) expresses the *energy theorem*:

$$E(T_2) = E(T_1) + L_p + L_q + L_f + L_v . \tag{3.3}$$

In (3.3), $L_f = \int_\Delta y_t \, 2f \, d\Delta$ is the work of active forces, and

$$L_v = \int_{\Lambda \cap \Delta} (y_t^+ + y_t^-) \, \Omega'(t) \, dt$$

is the work of the unilateral constraint, obtained as a product of the reaction $2\,\Omega'$ by a half of the sum of the speeds before and after the impact. In particular, if $\lambda' \equiv 0$, then $y_t^+ = y_t^-$ a.e. and we obtain

$$L_v = 2 \int_{\Delta \cap \Lambda} \beta'(t) \, \Omega'(t) \, dt \tag{3.4}$$

since in the points where $y_t \neq \beta'$ it is $\Omega' = 0$.

As a *corollary*, if the active forces are missing ($f = 0$), and the constraints are fixed ($A', B', p', q', \lambda', \beta' = 0$), we obtain the *principle of conservation of the energy*: $E(T_2) = E(T_1)$.

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