Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Luigi Amerio

On the motion of a string vibrating through a moving ring with a continuously variable diameter

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.2, p. 134–142.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_2_134_0>

Analisi matematica. — *On the motion of a string vibrating through a moving ring with a continuously variable diameter.* Nota [(*)] del Corrisp. LUIGI AMERIO [(**)].

RIASSUNTO. — Si studia il moto di una corda vibrante, vincolata ad attraversare un anello ortogonale al piano di vibrazione, con centro e diametro variabili comunque nel tempo. La soluzione viene ricondotta a quella di un *problema elementare* che viene risolto esplicitamente.

1. Consider the vibrating string equation, in the *characteristic form* (and in the sense of *distributions*):

$$y_{\xi\eta} = f(\xi, \eta) = f(\mathrm{P}) \tag{1.1}$$

where $\xi = (x + t)\, 2^{-\frac{1}{2}}$, $\eta = (-x + t)\, 2^{-\frac{1}{2}}$. In (1.1) $2f(\mathrm{P})$ denotes the external force, $y(\mathrm{P})$ is the displacement from the x axis, $t \geq 0$ is the time. We assume that the string, at rest, is placed on the x axis. We assume moreover that the free vibration of the string, in the (x, y) plane, is impeded by a *ring* through which the string is obliged to pass: such a ring is always orthogonal to the plane (x, y), has the center in the point $\mathrm{G}(\lambda(t), (\alpha(t) + \beta(t))/2)$ and the diameter $\beta(t) - \alpha(t)$; $\alpha(t)$ and $\beta(t) \geq \alpha(t)$ are arbitrary *continuous* functions, $\lambda(t)$ satisfies only the *Lipschitz condition* $|\lambda'(t)| \leq 1$ a.e., never being $\lambda'(t) = \pm 1$ on an interval. This means that the longitudinal velocity of the ring cannot be greater than the velocity of a wave traveling in the string: moreover the equality does not hold on an interval.

The problem considered has the following analytical interpretation. We consider, in the (x, t) plane, a line Λ, $x = \lambda(t)$, and impose that the displacement $y(x, t)$ satisfies the following *pair of unilateral conditions*:

$$\alpha(t) \leq y(\lambda(t), t) \leq \beta(t) \qquad (t \geq 0). \tag{1.2}$$

Observe that if $\alpha(t) \equiv \beta(t)$ on an interval $a \vdash\!\dashv b$, (1.2) implies a kind of *boundary condition*, since we impose the value of the displacement on $a \vdash\!\dashv b$: $y(\lambda(t), t) = \alpha(t)$. If $\alpha(t) = -\infty$, or $\beta(t) = +\infty$, the preceding problem has been solved in [1] (for another type of unilateral problems cfr. [2], [3]).

The solution will be obtained by reducing it, as in the other cases, to that of *elementary problems*: to those of Cauchy, Darboux and Goursat, we must add now another problem, that we shall call $\Pi_{\alpha\beta}$ problem, by generalising the Π problem solved in [1]. By solving, at § 2, the $\Pi_{\alpha\beta}$ problem, we can calculate the *reaction of the ring*.

(*) Presentata nella seduta del 12 febbraio 1977.
(**) Istituto matematico del Politecnico di Milano.

As we shall prove, *the problem considered has one, and only one, solution*, without imposing any condition on the nature of the impact against the obstacle (elastic, partially elastic, anelastic): this makes a notable difference from the case of the impact against a wall (cfr. [2], [3]).

2. Let us define, on the rectangle $R = OLNH = \{0 \le \xi \le l, 0 \le \eta \le h\}$ of the (ξ, η) plane (fig. 1), the following $\Pi_{\alpha\beta}$ *problem*.

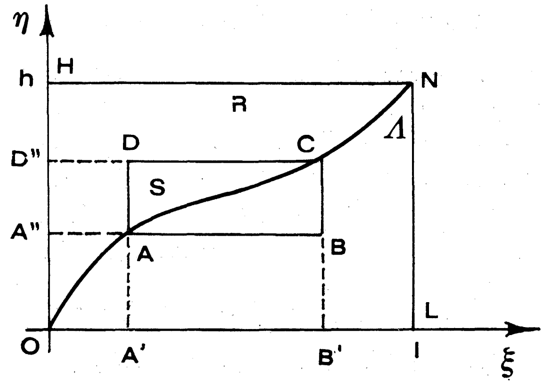

Fig. 1.

Let Λ be a line of equation

$$\eta = g(\xi), \qquad 0 \le \xi \le l, \tag{2.1}$$

where $g(\xi)$ is a *continuous, strictly increasing function*, $g(0) = 0$, $g(l) = h$. Let moreover $\alpha(P)$ and $\beta(P)$ be two *continuous* functions, defined on Λ and such that

$$\alpha(P) \le \beta(P) \quad \forall P \in \Lambda \quad , \quad \alpha(0) \le 0 \le \beta(0). \tag{2.2}$$

We claim to calculate, on R, *a function* $\Gamma(P)$ *which satisfies the following conditions* ($\Pi_{\alpha\beta}$ *problem*):

(2.3) 1) $\Gamma(P) \in C^0(R)$,

2) $\Gamma(P) = 0$ *on* $OL \cup OH$,

3) $\alpha(P) \le \Gamma(P) \le \beta(P) \quad \forall P \in \Lambda$,

4) $\operatorname{Supp} \Gamma_{\xi\eta} \subseteq \{P \in \Lambda : \Gamma(P) = \alpha(P) \quad \textit{or} \quad \Gamma(P) = \beta(P)\}$,

5) $\Gamma_{\xi\eta} \ge 0$ on every arc $\Lambda' \subseteq \mathring{\Lambda}$ where $\Gamma(P) < \beta(P)$,

$\Gamma_{\xi\eta} \le 0$ on every arc $\Lambda'' \subseteq \mathring{\Lambda}$ where $\Gamma(P) > \alpha(P)$.

In 4) and 5) the derivative $\Gamma_{\xi\eta}$ is obviously a *distribution* $\in \mathcal{D}'(\mathring{R})$; by 4), $\Gamma(P)$ satisfies the homogeneous equation $\Gamma_{\xi\eta} = 0$ on the open set $\mathring{R} - \operatorname{Supp} \Gamma_{\xi\eta} \supseteq \mathring{R} - \Lambda$.

We shall prove that $\Pi_{\alpha\beta}$ *problem has one, and only one, solution.*

a) Let us observe, first of all, that 1), 4) and 5) imply (fig. 1) the condition

$$\Gamma(C) - \Gamma(D) - \Gamma(B) + \Gamma(A) \geq 0\,, \tag{2.4}$$

for every rectangle $S = ABCD$, with the edges respectively parallel to the ξ and η axis and the vertices A and C on a Λ' arc. If $A, C \in \Lambda''$, we have

$$\Gamma(C) - \Gamma(D) - \Gamma(B) + \Gamma(A) \leq 0\,. \tag{2.5}$$

b) *Uniqueness.* Assume that there exist two solutions, $\Gamma(P)$ and $\tilde{\Gamma}(P)$. By 4), $\tilde{\Gamma}(P) \equiv \Gamma(P)$ on $\Lambda \Rightarrow \tilde{\Gamma}(P) \equiv \Gamma(P)$ on R: hence it is sufficient to prove that $\tilde{\Gamma}(P) \equiv \Gamma(P)$ on Λ. Assume the contrary, that is $\tilde{\Gamma}(P_0) > \Gamma(P_0)$, where $P_0 \in \Lambda$. Since $\tilde{\Gamma}(0) = \Gamma(0) = 0$, there exists (fig. 1) an arc $\widehat{AC} \ni P_0$ such that $\tilde{\Gamma}(A) = \Gamma(A)$ and $\tilde{\Gamma}(P) > \Gamma(P) \geq \alpha(P)$ for $A < P \leq C$ (on Λ). We have, by 5), (2.5) and 2),

$$\begin{aligned} \tilde{\Gamma}(C) - \tilde{\Gamma}(A) &\leq (\tilde{\Gamma}(B) - \tilde{\Gamma}(A)) + (\tilde{\Gamma}(D) - \tilde{\Gamma}(A)) = \\ &= (\tilde{\Gamma}(B') - \tilde{\Gamma}(A')) + (\tilde{\Gamma}(D'') - \tilde{\Gamma}(A'')) = 0\,. \end{aligned} \tag{2.6}$$

We have moreover, since $\Gamma(P) < \tilde{\Gamma}(P) \leq \beta(P)$ for $A < P \leq C$,

$$\Gamma(C) - \Gamma(A) \geq 0\,. \tag{2.7}$$

Hence $\Gamma(C) \geq \Gamma(A) = \tilde{\Gamma}(A) \geq \tilde{\Gamma}(C)$, which is absurd.

c) *Existence.* In order to prove the existence of the solution $\Gamma(P)$, we shall obtain, at first, the *trace* of $\Gamma(P)$ on Λ, by assuming that it is

$$\alpha(P) < \beta(P) \qquad \forall P \in \mathring{\Lambda}\,, \tag{2.8}$$

that is for $0 < P < N$. Let us set moreover, in order to simplify notations,

$$\begin{aligned} &\alpha(\xi) = \alpha(\xi, g(\xi)) \quad , \quad \beta(\xi) = \beta(\xi, g(\xi)) \\ &\Omega(\xi) = \Omega(\xi, g(\xi)) = \Gamma(\xi, g(\xi))\,. \end{aligned} \tag{2.9}$$

We have $\alpha(0) \leq 0 = \Omega(0) \leq \beta(0)$ and let $\xi_0 \geq 0$ be the greatest value of ξ such that $\alpha(t) \leq 0$ and $\beta(t) \geq 0$ $\forall t \leq \xi$: it cannot be $\alpha(\xi_0) < 0$ and $\beta(\xi_0) > 0$. Set

$$\Omega(\xi) = 0 \qquad \text{on} \quad 0 \vdash\!\dashv \xi_0 \; (\Rightarrow \alpha(\xi) \leq \Omega(\xi) \leq \beta(\xi)) \tag{2.10}$$

and consider the interval $\xi_0 \vdash\!\dashv l$. There are three possibilities:

$$\begin{aligned} &\text{I)} \quad \alpha(\xi_0) = 0 \quad \text{and} \quad \beta(\xi_0) > 0\,, \\ &\text{II)} \quad \alpha(\xi_0) < 0 \quad \text{and} \quad \beta(\xi_0) = 0\,, \\ &\text{III)} \quad \alpha(\xi_0) = 0 \quad \text{and} \quad \beta(\xi_0) = 0\,. \end{aligned} \tag{2.11}$$

Let us consider the case I): in the same way we can discuss II). Let us define, on the triangle $\xi_0 \leq \tau \leq \xi \leq l$ in the plane (τ, ξ), the following functions:

$$\phi(\tau, \xi) = \max_{\tau \leq t \leq \xi} \alpha(t) \quad , \quad \psi(\tau, \xi) = \min_{\tau \leq t \leq \xi} \beta(t) . \tag{2.12}$$

Since, by I),

$$\phi(\xi_0, \xi_0) = \alpha(\xi_0) = \Omega(\xi_0) < \beta(\xi_0) ,$$

there exists an interval $\xi_0 \vdash\!\dashv \xi_1$ (and we shall take the largest interval) such that

$$\phi(\xi_0, \xi) \leq \beta(\xi) \qquad \text{on} \quad \xi_0 \vdash\!\dashv \xi_1 .$$

Set (fig. 2)

$$\Omega(\xi) = \phi(\xi_0, \xi) \qquad \text{on} \quad \xi_0 \vdash\!\dashv \xi_1 \; (\Rightarrow \Omega(\xi_1) = \beta(\xi_1) , \;\; \text{if} \;\; \xi_1 < l) .$$

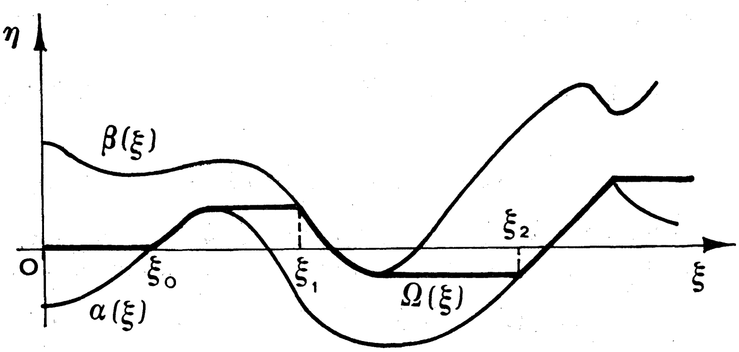

Fig. 2.

If $\xi_1 < l$, it is $\beta(\xi_1) > \alpha(\xi_1)$ and we can consider the largest interval $\xi_1 \vdash\!\dashv \xi_2$ such that

$$\psi(\xi_1, \xi) \geq \alpha(\xi) \qquad \text{on} \quad \xi_1 \vdash\!\dashv \xi_2 .$$

Set

$$\Omega(\xi) = \psi(\xi_1, \xi) \qquad \text{on} \quad \xi_1 \vdash\!\dashv \xi_2 \; (\Rightarrow \Omega(\xi_2) = \alpha(\xi_2) , \; \text{if} \; \xi_2 < l) .$$

In such a way, we may attain the value l in a finite number of steps, or we obtain a sequence $\{\xi_n\}$, where $\xi_n < \xi_{n+1} < l$, such that

$$\begin{aligned} \Omega(\xi) &= \phi(\xi_{2n}, \xi) \qquad \text{on} \quad \xi_{2n} \vdash\!\dashv \xi_{2n+1} \\ \Omega(\xi) &= \psi(\xi_{2n+1}, \xi) \qquad \text{on} \quad \xi_{2n+1} \vdash\!\dashv \xi_{2n+2} . \end{aligned} \tag{2.13}$$

Setting

$$\lim_{n \to \infty} \xi_n = \bar{\xi} \leq l ,$$

it cannot be $\bar{\xi} < l$. If so, we have in fact $\alpha(\bar{\xi}) < \beta(\bar{\xi}) , \Rightarrow \alpha(\xi') < \beta(\xi'')$, $\forall \xi'$ and ξ'' belonging to an interval $\rho \vdash\!\dashv \bar{\xi}$: hence $\phi(\rho, \xi) < \beta(\xi)$ on $\rho \vdash\!\dashv \bar{\xi}$.

Since $\xi_{2n} \geq \rho$ for $n \geq n_\rho$, we have then

$$\beta(\xi_{2n+1}) = \Omega(\xi_{2n+1}) = \phi(\xi_{2n}, \xi_{2n+1}) \leq \phi(\rho, \xi_{2n+1}) < \beta(\xi_{2n+1}),$$

which is absurd. Therefore $\bar{\xi} = l$, and setting

$$\Omega(l) = \beta(l) = \lim_{n\to\infty} \phi(\xi_{2n}, \xi_{2n+1}) = \alpha(l),$$

the function $\Omega(\xi)$ is continuous on all $0 \vdash\!\dashv l$.

Consider now the case III): we have, necessarily,

$$\phi(\xi_0, \xi) > 0, \qquad \text{or} \quad \phi(\xi_0, \xi) < 0 \qquad \forall \xi \in \xi_0 \!-\!\!\dashv l. \tag{2.14}$$

Assume $\phi(\xi_0, \xi) > 0$. Let then ρ_n be the least value belonging to the interval $\xi_0 \!-\!\!\dashv \xi_0 + 1/n$, such that

$$\alpha(\rho_n) = \phi\left(\xi_0, \xi_0 + \frac{1}{n}\right) \; (\Rightarrow \alpha(\rho_n) > 0 ; n = 1, 2, \cdots). \tag{2.15}$$

Let moreover $\Omega_n(\xi)$ be the function defined on $\rho_n \vdash\!\dashv l$ as in the case I), with the initial value

$$\Omega_n(\rho_n) = \alpha(\rho_n) = \phi(\rho_n, \rho_n). \tag{2.16}$$

We shall set therefore $\Omega_n(\xi) = \phi(\rho_n, \xi) = \phi(\xi_0, \xi)$ on the largest interval where $\phi(\rho_n, \xi) \leq \beta(\xi)$, and so on. It is $\rho_{n+1} \leq \rho_n$, and we shall prove that

$$\Omega_{n+1}(\xi) = \Omega_n(\xi) \qquad \forall \xi \in \rho_n \vdash\!\dashv l. \tag{2.17}$$

We have in fact, by the construction made,

$$\Omega_{n+1}(\rho_n) \geq \alpha(\rho_n) = \Omega_n(\rho_n) = \phi(\xi_0, \rho_n)$$

and moreover, on all $\rho_{n+1} \vdash\!\dashv l$,

$$\Omega_{n+1}(\xi) \leq \phi(\xi_0, \xi), \Rightarrow \Omega_{n+1}(\rho_n) \leq \phi(\xi_0, \rho_n).$$

Hence $\Omega_{n+1}(\rho_n) = \Omega_n(\rho_n)$, $\Rightarrow$ (2.17). We set then

$$\begin{aligned} &\Omega(\xi) = \lim_{n\to\infty} \Omega_n(\xi) && \text{on} \quad \xi_0 \!-\!\!\dashv l, \\ &\Rightarrow \Omega(\xi) \in C^0(0 \vdash\!\dashv l) && \text{and} \quad \alpha(\xi) \leq \Omega(\xi) \leq \beta(\xi). \end{aligned} \tag{2.18}$$

The function $\Omega(\xi)$ has been therefore defined on $0 \vdash\!\dashv l$, assuming $\alpha(\xi) < \beta(\xi)$ on $0 \!-\!\!- l$. In the general case, we shall set $\Omega(\bar{\xi}) = \alpha(\bar{\xi}) = \beta(\bar{\xi})$ at every point $\bar{\xi}$ where $\alpha = \beta$, and we shall define $\Omega(\xi)$ as before on the open intervals of the complementary set.

Let us give, at last, the *solution* $\Gamma(P)$ *of the problem* $\Pi_{\alpha\beta}$.

We have $\forall P \in \Lambda$, $\Omega(P) = \Omega(\xi, g(\xi)) = \Omega(\xi)$, and we set moreover

$$\Gamma(\xi, \eta) = \begin{cases} \Omega(\xi, g(\xi)) & \text{for} \quad g(\xi) \leq \eta \leq h \\ \Omega(g^{-1}(\eta), \eta) & \text{for} \quad g^{-1}(\eta) \leq \xi \leq l, \end{cases} \tag{2.19}$$

where g^{-1} is the inverse function of g. It is obvious that $\Gamma(\xi, \eta)$ satisfies conditions 1), 2), 3). We have, moreover, $\Gamma_{\xi\eta} = 0$ on $\mathring{R} - \Lambda$, $\Rightarrow \operatorname{Supp} \Gamma_{\xi\eta} \subseteq \Lambda$.

Assume now $\alpha(P_0) < \Omega(P_0) < \beta(P_0)$, $P_0 \in \mathring{\Lambda}$: there exists then, by the definition of $\Omega(\xi)$, an arc $\widehat{AC} \subset \Lambda$, with $P_0 \in \widehat{AC}$, such that $\Omega(P) \equiv \Omega(P_0)$ $\forall P \in \widehat{AC}$. We have then, by (2.19), $\Gamma(P) \equiv \Gamma(P_0)$ on all the corresponding rectangle $S = ABCD$ (fig. 1): hence $P_0 \notin \operatorname{Supp} \Gamma_{\xi\eta}$, and condition 4) holds.

Assume, at last, $\Gamma(P) < \beta(P)$ on the arc $AC = \Lambda' \subset \mathring{\Lambda}$. We have, by (2.19), $\Gamma(B) = \Gamma(A)$, $\Gamma(D) = \Gamma(A)$: therefore

$$\Gamma(C) - \Gamma(B) - \Gamma(D) + \Gamma(A) = \Gamma(C) - \Gamma(A) \geq 0,$$

since, by the construction made, $\Omega(P)$ is an increasing function on $\widehat{AC}$. We have then $\Gamma_{\xi\eta} \geq 0$ on Λ', $\Rightarrow$ 5).

Observation I. – *Assume* $\beta = +\infty$ (*or* $\alpha = -\infty$). The trace $\Omega(\xi)$ has then a very simple form (cfr. [1]):

$$\Omega(\xi) = \max_{0 \vdash\!\dashv \xi} \alpha^{+}(t) \qquad (\text{or } \Omega(\xi) = \min_{0 \vdash\!\dashv \xi} \beta^{-}(t)). \tag{2.20}$$

Observation II. – We can generalise $\Pi_{\alpha\beta}$ problem if we substitute the condition 2) by the condition:

$$2') \quad \Gamma(P) = \zeta(P) \qquad \textit{on} \quad OL \cup OH, \tag{2.21}$$

where $\zeta(P) \in C^0(OL \cup OH)$ and satisfies only the (necessary) condition

$$\alpha(0) \leq \zeta(0) \leq \beta(0). \tag{2.22}$$

Setting $P = (\xi, \eta)$, $P' = (\xi, 0)$, $P'' = (0, \eta)$, and moreover

$$\bar{\Gamma}(P) = \Gamma(P) - \{\zeta(P') + \zeta(P'') - \zeta(0)\}, \tag{2.23}$$

we have $\bar{\Gamma}_{\xi\eta} = \Gamma_{\xi\eta}$, and $\bar{\Gamma}(P)$ satisfies condition 2). Let us now calculate $\bar{\Gamma}(P)$ by imposing conditions 1), 2), 3), 4), 5), where $\alpha(P)$ and $\beta(P)$ are substituted by $\alpha(P) - \zeta(P') - \zeta(P'') + \zeta(0)$ and by $\beta(P) - \zeta(P') - \zeta(P'') + \zeta(0)$. Therefore $\bar{\Gamma}(P)$ exists and is unique: we define afterwards $\Gamma(P)$ by (2.23) and we have:

$$\begin{aligned} \operatorname{Supp} \Gamma_{\xi\eta} = \operatorname{Supp} \bar{\Gamma}_{\xi\eta} \subseteq \{P \in \Lambda : \bar{\Gamma}(P) = \alpha(P) - \zeta(P') - \zeta(P'') + \zeta(0) \quad & \textit{or} \\ \bar{\Gamma}(P) = \beta(P) - \zeta(P') - \zeta(P'') + \zeta(0)\} = & \\ = \{P \in \Lambda : \Gamma(P) = \alpha(P) \quad \textit{or} \quad \Gamma(P) = \beta(P)\}. & \end{aligned}$$

It is moreover $\Gamma_{\xi\eta} = \bar{\Gamma}_{\xi\eta} \geq 0$ on every arc Λ' where $\bar{\Gamma}(P) < \beta(P) - \zeta(P') - \zeta(P'') + \zeta(0)$, that is $\Gamma(P) < \beta(P)$, and analogously for the opposite inequality.

We conclude that *problem* $\Pi_{\alpha\beta}$, *with conditions* 1), 2'), 3), 4), 5) *has one, and only one, solution*: $\Gamma(P) = \bar{\Gamma}(P) + \zeta(P') + \zeta(P'') - \zeta(0)$.

3. We apply now the preceding results in order to solve the problem described at § 1 (cfr. [1]). Let us consider equation (1.1) in a domain Z of the (x, t) plane, defined by the inequalities:

$$t \geq 0 \quad , \quad p(t) \leq x \leq q(t) , \tag{3.1}$$

where $p(t)$ and $q(t)$ satisfy Lipschitz conditions, and $p(t) < q(t)\ \forall t$. We assume moreover $|p'(t)| \leq 1$, $|q'(t)| \leq 1$ a.e., without being $p'(t) = \pm 1$, or

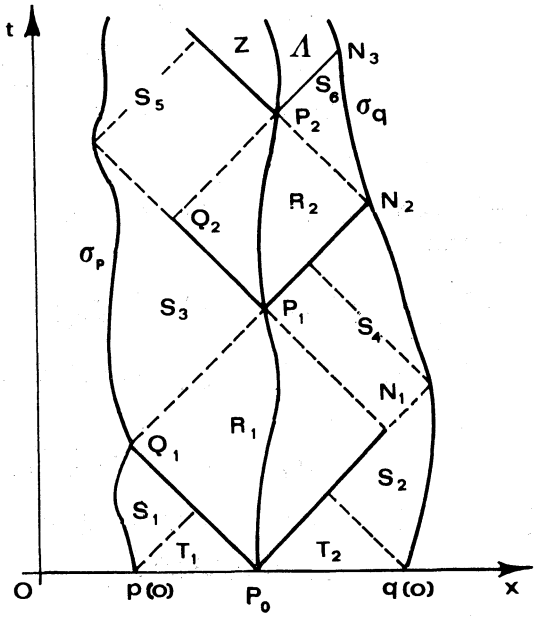

Fig. 3.

$q'(t) = \pm 1$, on an interval: therefore we exclude that the boundary lines $(\sigma_p = \{x = p(t)\}, \sigma_q = \{x = q(t)\})$ contain any characteristic segment.

Suppose that there are assigned the Cauchy *initial conditions*:

$$y(x, 0) = \varphi(x) \quad , \quad y_t(x, 0) = \psi(x) \qquad (p(0) \leq x \leq q(0)) \tag{3.2}$$

and the *boundary conditions*:

$$y(p(t), t) = A(t) \quad , \quad y(q(t), t) = B(t) \qquad (t \geq 0) . \tag{3.3}$$

Consider now, in Z, a *line* Λ, $x = \lambda(t)$, where $\lambda(t)$ satisfies the same type of conditions as $p(t)$ and $q(t)$, and it is (fig. 3):

$$p(t) < \lambda(t) < q(t) \qquad \forall t \geq 0 . \tag{3.4}$$

We impose that *the displacement* $y(x, t)$ *satisfies* (1.1) *on* $\mathring{Z} - \Lambda$; *moreover* $y(x, t)$ *must satisfy the initial and boundary conditions, and the inequalities*

$$\alpha(P) \leq y(P) \leq \beta(P) \qquad \forall P \in \Lambda . \tag{3.5}$$

The *data* are supposed to satisfy the following hypotheses:

(3.6) a) $\varphi'(x), \psi(x) \in L^1(p(0) \vdash\!\dashv q(0))$,

b) $A(t), B(t) \in C^0(0 \vdash +\infty)$, $A(0) = \varphi(p(0))$, $B(0) = \varphi(q(0))$,

c) $\alpha(P), \beta(P) \in C^0(\Lambda)$, $\alpha(P_0) \leq \varphi(\lambda(0)) \leq \beta(P_0)$,

d) $f(P) \in L^1(Z_T) \qquad \forall T \geq 0$,

where $Z_T = \{0 \leq t \leq T, p(t) \leq x \leq q(t)\}$.

Let now W be the functional class defined by the conditions:

$i_1)$ $w(P) \in C^0(Z)$,

$i_2)$ $w_\xi(P), w_\eta(P), w_{\xi\eta}(P) \in L^1(T_1 \cup T_2)$.

In such hypotheses *the free problem for* (1.1) has one, and only one, solution $u(x,t) \in W$. We may obtain $u(x,t)$ by a classical scheme: we solve, at first, the Cauchy problem in T_1 and in T_2; we solve, afterwards, the Darboux and Goursat problems in $S_1, S_2, R_1, S_3, S_4, R_2, \cdots$.

Let us consider now *the problem with the obstacle.* Such a problem can be reduced, as it is classical in *Mechanics*, to a free problem by introducing the *reaction of the obstacle*, obviously of impulse-type. In a precise way: *setting*

$$y(P) = u(P) + \Gamma(P), \tag{3.7}$$

we have to find a function $\Gamma(P)$ *defined by the following conditions*:

I) $\Gamma(P) \in C^0(Z)$,

II) $\Gamma(P) = 0$ *on the boundary* ∂Z,

III) $\alpha(P) - u(P) \leq \Gamma(P) \leq \beta(P) - u(P) \qquad$ *on* Λ,

IV) $\operatorname{Supp} \Gamma_{\xi\eta} \subseteq \{P \in \Lambda : \Gamma(P) = \alpha(P) - u(P),$ *or* $\Gamma(P) = \beta(P) - u(P)\}$,

V) $\Gamma_{\xi\eta} \geq 0 \qquad$ *on every arc* $\Lambda' \subset \mathring{\Lambda} \qquad$ *where* $\Gamma(P) < \beta(P)$,

$\Gamma_{\xi\eta} \leq 0 \qquad$ *on every arc* $\Lambda'' \subset \mathring{\Lambda} \qquad$ *where* $\Gamma(P) > \alpha(P)$,

VI) $\Gamma(P) = 0 \qquad$ *on* $T_1 \cup T_2 \cup S_1 \cup S_2$.

It is obvious that, if $\Gamma(P)$ satisfies conditions I), $\cdots$, VI), then the function $y(P)$ given by (3.7) will be a solution of our problem; observe that the condition VI) means that the impulses generated by the impact of the string against the obstacle do not influence the solution of the free problem at the exterior of the forward characteristic semicone with vertex in $P_0(\varphi(0), 0)$.

Let us prove that $\Gamma(P)$ *exists on all* Z *and is unique.* We know in fact $\Gamma(P)$, with value zero, on the lower edges, $P_0 N_1$ and $P_0 Q_1$ of the rectangle R_1: hence we can calculate $\Gamma(P)$ on R_1, by solving the $\Pi_{\alpha-u,\beta-u}$ problem. We obtain then $\Gamma(P)$ on $S_3 \cup S_4$, by solving the Darboux and Goursat problems for the equation $\Gamma_{\xi\eta} = 0$. Therefore the values of $\Gamma(P)$ are known on the edges $P_1 N_2$ and $P_1 Q_2$ of the rectangle R_2, and we can calculate $\Gamma(P)$ on all R_2 by solving problem $\Pi_{\alpha-u,\beta-u}$ (with non-zero values on the edges: § 2, observation II).

In such a way, we obtain the function $\Gamma(P)$ on all Z and (3.7) gives the *unique solution* $y(P)$ *of the problem*: *the reaction of the obstacle is the distribution* $\Gamma_{\xi\eta}$.

Let us observe, lastly, that, more generally, we can determine the motion of the string, in presence of more obstacles of the type considered before. In that case, one assumes that the domain Z contains *m lines* Λ_j, with the equations $x = \lambda_j(t)$, where $p(t) < \lambda_1(t) < \cdots < \lambda_m(t) < q(t)$. One assumes, moreover, that *the displacement* $y(P)$ *satisfies, on every* Λ_j, *the conditions*

$$\alpha_j(P) \leq y(P) \leq \beta_j(P).$$

It may be, for some j, $\alpha_j = -\infty$ or $\beta_j = +\infty$.

Also *this problem admits one, and only one, solution*, which can be obtained by extending, in an obvious way, the method described for $m = 1$.

REFERENCES

[1] L. AMERIO (1976) – *Su un problema di vincoli unilaterali per l'equazione non omogenea della corda vibrante*, IAC (Ist. per le applicazioni del calcolo «Mauro Picone»), Pubblicazioni, serie III, n. 109.

[2] L. AMERIO e G. PROUSE (1975) – *Study of the motion of a string vibrating against an obstacle*, «Rend. di Matematica» (2), *8*, serie VI.

[3] C. CITRINI (1975) – *Sull'urto parzialmente elastico o anelastico di una corda vibrante contro un ostacolo*, «Rend. Acc. Naz. dei Lincei», *49*, serie VIII.