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On a paper by Kuipers and Scheelbeek

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Teoria dei gruppi. — *On a paper by Kuipers and Scheelbeek.* Nota di JAU-SHYONG SHIUE (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Nel lavoro [2] si danno due risultati, qui richiamati come Teoremi A e B. In questa Nota si osserva mediante un controesempio che il Teor. A è incorretto e, ricorrendo ad un classico risultato di Eckmann [1], si stabilisce un Teorema 1, più generale del suddetto Teorema B.

In [1], Eckmann introduced the notion of uniform distribution of a sequence in a compact topological group: Let G be a compact topological group with Haar measure u with $u(G) = 1$. Let $\{x_n\}$ be a sequence from G, N be any positive integer. Put

$$A(M, N) = \sum_{\substack{n=1 \\ x_n \in M \subset G}}^{N} 1$$

Then $\{x_n\}$ is said to be uniformly distributed in G if, for each closed set M with boundary ∂M satisfying $u(\partial M) = 0$,

$$\lim_{N \to \infty} \frac{A(M, N)}{N}$$

exists and equals $u(M)$.

Eckmann [1] also proved the famous classical result.

If G *is a compact topological group, then the sequence* $\{x_n\}$ $(n = 1, 2, \cdots)$ *from* G *is uniformly distributed in* G *if, and only if, for each non-trivial irreducible representation* D *of* G

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} D(x_n) = 0. \tag{1}$$

If in addition G *is abelian, then* D *in* (1) *stands for a non-trivial character of* G.

In [2], Kuipers and Scheelbeek gave the following:

THEOREM A. *Let* G *and* H *be two compact topological groups. Let* $a \in G$ *be fixed point free under all non-trivial representations of* G. *Let* $b \in H$. *Then the sequence of elements* $(a, b), (a, b^2), (a^2, b), (a, b^3), (a^3, b), (a^2, b^2), (a^2, b^3), (a^3, b^2), (a, b^4), (a^4, b), \cdots$ *is uniformly distributed in* F, *the direct product of* G *and* H.

(*) This paper was written while the author was a Humboldt Stiftung fellow visiting University of Köln.

(**) Nella seduta del 12 febbraio 1977.

THEOREM B. *If* G *is an abelian group, and* H *is a normal subgroup of finite index, then the sequence* $\{x_n\}$ $(n = 1, 2, \cdots)$ *from* G *is uniformly distributed in the cosets of* G *mod* H *if, and only if,* (1) *holds for every non-trivial character* f *of* G *which is trivial on* H.

In this Note, we first remark that Theorem A is incorrect as can be seen by the following:

COUNTEREXAMPLE. Let $G = Z_2$ and $H = Z_4$ be the rings of integers mod 2 and mod 4 respectively in Theorem A with the discrete topology. Take $a = 1$, $b = 2$. Then the above mentioned sequence becomes

$$\{(1, 2), (1, 0), (1, 2), (1, 2), (1, 2), (1, 0), \cdots\}.$$

The sequence *is not uniformly distributed* in $Z_2 \times Z_4$, because $nb \equiv 0, 2 \pmod 4$.

Secondly, we want to prove

THEOREM 1. *If* G *is any topological group, and* H *is a normal subgroup of finite index, then the sequence* $\{x_n\}$ *from* G *is uniformly distributed in the right cosets (or left cosets) of* G *mod* H *if, and only if,* (1) *holds for every non-trivial irreducible representation* D *of* G *which is trivial in* H.

Proof. We know by definition that $\{x_n\}$ is uniformly distributed in the right cosets X of G mod H if, and only if,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ x_n \in X}}^{N} 1 = \{G : H\}^{-1} \qquad \text{for all } X. \tag{2}$$

On the other hand (2) holds if, and only if,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ x_n + H = X}}^{N} 1 = \{G : H\}^{-1} \qquad \text{for all } X. \tag{3}$$

Applying Eckmann's definition in the simplest form, (3) holds if, and only if,

$$\{x_n + H\} \qquad \text{is uniformly distributed in } G/H. \tag{4}$$

This means, by the classical Eckmann result, that (4) holds if, and only if,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D(x_n + H) = 0 \tag{5}$$

for all non-trivial irreducible representation D on G/H.

Moreover, the equation

$$D'(a) = D(a + H) \qquad \forall a \in G$$

implies that any representation on G/H can be induced as a representation on G which is trivial on H and conversely. Thus (5) holds if, and only if,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D'(x_n) = 0$$

for all non-trivial irreducible representation D′ on G which is trivial on H.

This proves Theorem 1 completely.

REMARK. We should like to point out another essential difference between our proof and Kuipers-Scheelbeek's. In Kuipers-Scheelbeek's paper [2], only rather elementary computations appear. Here we use Eckmann's theorem to prove Theorem 1 and make the argument shorter. Moreover, Theorem 1 is more general than that of Kuipers-Scheelbeek.

REFERENCES

[1] B. ECKMANN (1943-44) – *Ueber monothetische Gruppen*, «Comm. Math. Helv.», *16*, 249-263.

[2] L. KUIPERS and P. A. J. SCHEELBEEK (1968) – *Uniform distribution of sequences from direct products of groups*, «Ann. Sc. Norm. Sup. Pisa», *22*, 599-608.