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George Bachman, Pao-sheng Hsu

Extension of lattice continuous maps to generalized Wallman spaces

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Extension of lattice continuous maps to generalized Wallman spaces.* Nota di GEORGE BACHMAN e PAO-SHENG HSU, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si introduce la nozione di applicazioni reticolarmente continue (lattice continuous maps) e si danno condizioni sotto le quali esse possono essere estese agli spazi di Wallman generalizzati associati. I teoremi dati generalizzano le WC-applicazioni di D. Harris ed estendono ad un « lattice setting » generale un importante teorema di Arhangel'skii.

1. INTRODUCTION

Let X be an *arbitrary* set and $\mathscr{L}$ a lattice of subsets. Under suitable conditions (section 2) on $\mathscr{L}$, there is associated a compact T_1 space, $I_R(\mathscr{L})$, consisting of those two-valued lattice regular measures defined on the algebra generated by $\mathscr{L}$ and suitably topologized. Kerner [6] has developed various properties of $I_R(\mathscr{L})$; other work has been done in [2], [9], [7], and [3], generalizing much of the work of Frink [4] and others, which involves a normal base lattice in a completely regular T_2 space.

It is our intention here to investigate when, given two sets X and Y with lattice $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively and a map $T : X \to Y$ which is lattice continuous (see section 3), the map T can be extended to a continuous map of $I_R(\mathscr{L}_1)$ into $I_R(\mathscr{L}_2)$. When applied to specific topological spaces X and Y and specific lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ therein, the general theorems developed here will yield as special cases important theorems of Harris [5] and Arhangel'skii [8].

We first give in section 2 the relevant notations, definitions, and critical theorems used so as to make the paper reasonably self-contained. Further

(*) Nella seduta del 12 febbraio 1977.

details can be found in the references. In section 3, we will prove the main theorems on extensions and give several applications. It should be clear from the material developed that more applications as well as useful filter interpretation can be given.

2. Background and Notations

In general, X is a set and $\mathscr{L}$ a lattice of subsets of X with X and ϕ in $\mathscr{L}$. A lattice $\mathscr{L}$ is *disjunctive* if, for $x \in X$, $x \notin L_1 \in \mathscr{L}$, there exists $L_2 \in \mathscr{L}$ such that $x \in L_2$ and $L_1 \cap L_2 = \phi$. A lattice $\mathscr{L}$ is *separating* if, for all x_1 and x_2 in X, $x_1 \neq x_2$, there exists $L \in \mathscr{L}$ such that $x_1 \in L$ and $x_2 \in L'$ (the complement of L).

We shall consider the set $I_R(\mathscr{L})$ of all $\mathscr{L}$-regular two-valued finitely additive measures on the smallest algebra $\mathscr{A}(\mathscr{L})$ containing $\mathscr{L}$ (i.e., $\mu(A) = \sup_{\substack{L \subset A \\ L \in \mathscr{L}}} \mu(L)$, for any measure $\mu \in I_R(\mathscr{L})$). Extending the correspondence between two-valued measures and ultrafilters on a Boolean algebra, Bachman and Cohen [2] established the one-to-one correspondence between all two-valued $\mathscr{L}$-regular measures on $\mathscr{A}(\mathscr{L})$ and all $\mathscr{L}$-ultrafilters: Given an $\mathscr{L}$-ultrafilter $\mathscr{F}$, $\mathscr{A}(\mathscr{F}) = \{B \in \mathscr{P}(X) : B \supseteq A$ for some $A \in \mathscr{F}$ or $B' \supseteq A$ for some $A \in \mathscr{F}\}$ is an algebra containing $\mathscr{L}$ with an $\mathscr{L}$-regular measure $\mu_{\mathscr{F}}$ defined on $\mathscr{A}(\mathscr{F})$, where

$$\mu_{\mathscr{F}}(B) = \begin{cases} 1, & \text{if } B \supseteq A, A \in \mathscr{F} \\ 0, & \text{if } B' \supseteq A, A \in \mathscr{F}. \end{cases}$$

Conversely, given an $\mathscr{L}$-regular measure μ on $\mathscr{A}(\mathscr{L})$, the set $\mathscr{F} = \{A \in \mathscr{L} : \mu(A) = 1\}$ is an $\mathscr{L}$-ultrafilter.

Some properties of the lattice $\mathscr{L}$ are reflected on such measures.

A lattice $\mathscr{L}$ is *normal* if, for all disjoint L_1 and L_2 in $\mathscr{L}$, there exists L_3 and L_4 in $\mathscr{L}$ such that $L_1 \subseteq L_3'$, $L_2 \subseteq L_4'$, and $L_3' \cap L_4' = \phi$. If $\mathscr{L}$ is normal with an arbitrary two-valued measure μ on $\mathscr{A}(\mathscr{L})$, and there are two-valued $\mathscr{L}$-regular measures γ_1 and γ_2 on $\mathscr{A}(\mathscr{L})$ such that $\mu \leq \gamma_1$ and $\mu \leq \gamma_2$ on $\mathscr{L}$, then it follows $\gamma_1 = \gamma_2$. Conversely, suppose for every measure μ on $\mathscr{A}(\mathscr{L})$, there is a unique $\mathscr{L}$-regular measure τ on $\mathscr{A}(\mathscr{L})$ such that $\mu(L) \leq \tau(L)$ for every $L \in \mathscr{L}$ (this is to be denoted from now on by $\mu \leq \tau(\mathscr{L})$), then $\mathscr{L}$ is normal: Otherwise, there exist L_1 and L_2 in $\mathscr{L}$ with $L_1 \cap L_2 = \phi$, and the set $\mathscr{H} = \{L' \in \mathscr{L}' : L_1 \subseteq L'$ or $L_2 \subseteq L'\}$ has the finite intersection property, with a two-valued measure μ on $\mathscr{A}(\mathscr{L})$ such that $\mu(L') = 1$ for all $L' \in \mathscr{H}$ [7]. Now for any L in $\mathscr{L}$ with $\mu(L) = 1$ or $\mu(L') = 0$, we have $L_1 \nsubseteq L'$ and $L_2 \nsubseteq L'$, or, $L_1 \cap L = \phi$ and $L_2 \cap L = \phi$. Then there are $\mathscr{L}$-regular measures τ_1 and τ_2 on $\mathscr{A}(\mathscr{L})$ such that $\tau_1(L_1) = 1$, $\mu \leq \tau_1(\mathscr{L})$, $\tau_2(L_2) = 1$, and $\mu \leq \tau_2(\mathscr{L})$. Since $L_1 \cap L_2 = \phi$, $\tau_1 \neq \tau_2$ [7].

A lattice $\mathscr{L}$ is *compact* or the set X is *$\mathscr{L}$-compact* if, for every covering $\{L'_\alpha : L_\alpha \in \mathscr{L}\}$ of X there exists a finite subcover, $X = \bigcup_{\text{fin}} L'_\alpha$. The set X is $\mathscr{L}$-compact if and only if $\cap \{L_\alpha \in \mathscr{L} : \mu(L_\alpha) = 1$, where $\mu \in I_R(\mathscr{L})\} \neq \phi$.

Suppose that $(X, \mathscr{T})$ is a topological space and $\mathscr{L}$ a base for the closed sets $\mathscr{F}$ in $\mathscr{T}$. Then X is $\mathscr{L}$-compact if and only if $(X, \mathscr{T})$ is compact. In particular, if X is a set with a lattice $\mathscr{L}$ of subsets, let $\mathscr{T} = \tau(\mathscr{L})$ be the topology on X induced by using $\mathscr{L}$ as a base for the closed sets, then $(X, \tau(\mathscr{L}))$ is compact if and only if X is $\mathscr{L}$-compact [1].

In the case that X is a T_2 completely regular space and the lattice $\mathscr{L}$ of zero sets is normal, it was shown [2] that the set $I_R(\mathscr{L})$, given the Wallman-Frink type topology, is a generalized Frink type T_2 compactification. More generally, Koltun [7], Sultan [9], and Kerner [6] considered $I_R(\mathscr{L})$ for an arbitrary lattice $\mathscr{L}$ of subsets of a set X. The relevant results are summarized in the following:

PROPOSITION. *Let $\mathscr{L}$ be a lattice of subsets of a set* X, *with* X *and* ϕ *in* $\mathscr{L}$, *and* $\mathscr{L}$ *be disjunctive and separating. Let* $I_R(\mathscr{L})$ *be the set of all* $\mathscr{L}$*-regular two-valued measures on* $\mathscr{A}(\mathscr{L})$ *with the topology* O_w *in which the set* W (L) *is* $\{\mu \in I_R(\mathscr{L}) : \mu(L) = 1\}$ *for any* $L \in \mathscr{L}$ *and the collection* $\mathscr{W} = \{W(L) : L \in \mathscr{L}\}$ *is a base for the closed sets. Then* $(I_R(\mathscr{L})\, O_w)$ *is a* T_1 *compactification of* X, *where* X *has the topology induced by using* $\mathscr{L}$ *as a base for the sets.*

Proof. 1) $(I_R(\mathscr{L}), O_w)$ is T_1. Suppose μ_1 and μ_2 are distinct elements in $I_R(\mathscr{L})$, there exists $L \in \mathscr{L}$ such that $\mu_1(L) = 1$ and $\mu_2(L') = 1$. By $\mathscr{L}$-regularity, there exists $D \in \mathscr{L}$, $D \subseteq L'$, such that $\mu_2(D) = 1$; also $L \cap D = \phi$ and $\mu_1(D) = 0$. Then $\mu_1 \in W(D)'$, which does not contain μ_2.

2) $(I_R(\mathscr{L}), O_w)$ is compact: Suppose $\mathscr{H} = \{W(L), L \in \mathscr{L}\}$ is a family of basic closed sets with the finite intersection property. Then the family $\mathscr{K} = \{L : W(L) \in \mathscr{H}\}$ has the finite intersection property: For L_1 and L_2 in $\mathscr{L}$, $W(L_1 \cap L_2) = W(L_1) \cap W(L_2) \neq \phi$, and therefore $L_1 \cap L_2 \neq \phi$. The family $\mathscr{K}$ can be extended to a filter and then to an ultrafilter $\mathscr{F}$, which has an associated measure $\mu_{\mathscr{F}}$. For all $W(L) \in \mathscr{H}$, $L \in \mathscr{K} \subset \mathscr{F}$, and $\mu_{\mathscr{F}}(L) = 1$, or $\mu_{\mathscr{F}} \in W(L)$. Hence $\mu_{\mathscr{F}} \in \cap W(L)$. Thus the lattice $\mathscr{W}$ is compact. Therefore the space $(I_R(\mathscr{L}), O_w)$ is compact by an earlier observation.

3) We embed X in $I_R(\mathscr{L})$ by the mapping $\varphi : X \to I_R(\mathscr{L})$, $x \to \mu_x$ where μ_x is the $\mathscr{L}$-regular measure concentrated at the point x, defined by $\mu_x(A) = 1$ if and only if $x \in A$, for any $A \in \mathscr{Q}(\mathscr{L})$. The measure $\mu_x \in I_R(\mathscr{L})$ since $\mathscr{L}$ is dicjunctive. As $\mathscr{L}$ is separating, the map φ is one-to-one, from X onto $\mathscr{D} = \{\mu_x : x \in X\} \subset I_R(\mathscr{L})$. Let the set $\mathscr{D}$ have the relative topology induced by O_w. We would now identity $x \in X$ with μ_x in $I_R(\mathscr{L})$ and $L \in \mathscr{L}$ with the set $\{\mu_x \in I_R(\mathscr{L}) : x \in L\} = W(L) \cap \mathscr{D}$. If X has the topology $\tau(\mathscr{L})$ with $\mathscr{L}$ as the base for the closed sets, then it follows that the map φ is continuous $(\varphi^{-1}(W(L) \cap \mathscr{D}) = L)$ and closed $(\varphi(L) = W(L) \cap \mathscr{D})$, and the set $\mathscr{D}$ is dense in $I_R(\mathscr{L})$.

In addition, the compactification is Hausdorff if and only if the lattice $\mathscr{L}$ is normal [6], [7]. It has been pointed out [2] that if X is a Tyconoff space, $\mathscr{L} = \mathscr{Z}$, the δ-lattice of zero sets, then $I_R(\mathscr{L}) = \beta X$ is the Stone-Cech compactification; if X is Hausdorff zero-dimensional, $\mathscr{L} = \mathscr{P}$, the algebra of all clopen sets, then $I_R(\mathscr{L}) = \beta_0 X$ is the Banaschewski compactification; and if X is a T_1 space, $\mathscr{L} = \mathscr{F}$, the set of all closed sets, then $I_R(\mathscr{L}) = \omega(X)$ is the Wallman compactification.

3. Lattice Continuous Maps and Their Extensions

We consider mappings from X to Y, with $\mathscr{L}_1$ and $\mathscr{L}_2$ as lattices of subsets of X and Y respectively, where X and ϕ are in $\mathscr{L}_1$; Y and ϕ and in $\mathscr{L}_2$.

Definition 1. A map $T : X \to Y$ is said to be $\mathscr{L}_1$–$\mathscr{L}_2$ *continuous* if $T^{-1}(\mathscr{L}_2) \subseteq \mathscr{L}_1$.

In particular, if X and Y are given the topologies $\tau(\mathscr{L}_1)$ and $\tau(\mathscr{L}_2)$ respectively (where the lattice is the base for the closed sets), and if $T : X \to Y$ is $\mathscr{L}_1$–$\mathscr{L}_2$ continuous, then T is continuous topologically.

Maps which are $\mathscr{L}_1$–$\mathscr{L}_2$ continuous preserve some covering properties: Suppose $T : X \to Y$ is $\mathscr{L}_1$–$\mathscr{L}_2$ continuous. If X is $\mathscr{L}_1$-compact ($\mathscr{L}_1$-countably compact), then T (X) is compact (countably compact) relative to the lattice $T(X) \cap \mathscr{L}_2$.

Definition 2. A map $T : X \to Y$ is said to be $\mathscr{L}_1$–$\mathscr{L}_2$ *closed* if $T(\mathscr{L}_1) \subseteq \mathscr{L}_2$.

In the setting of the proposition in section 3, we should like to find conditions sufficient for an $\mathscr{L}_1$–$\mathscr{L}_2$ continuous map $T : X \to Y$ to be extended to a continuous map $\tilde{T} : I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$ on the compactification. We note that for each $\mu \in I_R(\mathscr{L}_2)$, since μT^{-1} is a measure on $\mathscr{A}(\mathscr{L}_2)$, there exists a $\gamma \in I_R(\mathscr{L}_2)$ such that $\mu T^{-1} \leq \gamma$ on $\mathscr{L}_2$ [7]. The measure γ is unique if $\mathscr{L}_2$ is normal.

I. *Generalized* WO *Maps*.

The next definition generalizes the setting of D. Harris [5]:

Definition 3. An $\mathscr{L}_1$–$\mathscr{L}_2$ continuous map $T : X \to Y$ is said to have the *generalized* WO *property* if for every covering $\{B_1', \cdots, B_n'\}$ of Y, where $B_i \in \mathscr{L}_2$, $i = 1, \cdots, n$, there exists $\{A_1, \cdots, A_m\} \subset \mathscr{L}_1$, such that: 1) $X = A_1' \cup \cdots \cup A_m'$, 2) for each A_k', $k = 1, \cdots, m$, and *any* $C \subseteq A_k'$, $C \in \mathscr{L}_1$, there exists a B_r' such that $\overline{T(C)} \subseteq B_r'$ (where closure is with respect to the $\tau(\mathscr{L}_2)$ topology).

Theorem 1. *The sets* X *and* Y *have lattices* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *respectively, both satisfying the conditions of the proposition in section 2. Suppose* $T : X \to Y$ *is* $\mathscr{L}_1$–$\mathscr{L}_2$ *continuous with the* WO *property and* $\overline{T(\mathscr{L}_1)} \subseteq \mathscr{L}_2$. *Then* T *can be extended to a continuous map* $\tilde{T} : I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$ *with* $\tilde{T}\mu = \gamma$, *where* γ *is the unique* $\mathscr{L}_2$*-regular measure such that* $\mu T^{-1} \leq \gamma(\mathscr{L}_2)$.

Proof. 1) The measure γ is unique: Suppose that there exists $\rho \in I_R(\mathscr{L}_2)$ with $\mu T^{-1} \leq \rho(\mathscr{L}_2)$ and $\gamma \neq \rho$. There are B_1 and B_2 in $\mathscr{L}_2$; $B_1 \cap B_2 = \phi$, $\gamma(B_1) = 1$, $\gamma(B_2) = 0$, $\rho(B_1) = 0$, and $\rho(B_2) = 1$. We have $Y = B_1' \cup B_2'$; by the WO property, there are $A_1, \cdots, A_m \in \mathscr{L}_1$, with $X = A_1' \cup \cdots \cup A_m'$. Hence there are A_k' with $\mu(A_k') = 1$ and $A \in \mathscr{L}_1$, $A \subseteq A_k'$, with $\mu(A) = 1$. Either $\overline{T(A)} \subseteq B_1'$ or $\overline{T(A)} \subseteq B_2'$. In the first case, $\gamma \overline{T(A)} \geq \mu T^{-1}(\overline{T(A)}) \geq \mu(A) = 1$, so $\gamma(B_1') = 1$: contradiction. The second case leads to a similar contradiction.

Define $\tilde{T} : I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$, where γ is the unique element in $I_R(\mathscr{L}_2)$ with

$\mu \to \gamma \geq T^{-1}$

$\gamma \geq \mu T^{-1}$ on $\mathscr{L}_2$. This map extends T: If $Tx = y$ and γ_x and γ_y are identified with x and y respectively, then $\tilde{T}\mu_x = \gamma_y$.

2) The map T is continuous: Let $\gamma \in I_R(\mathscr{L}_2)$ and $\gamma \in W(B)'$, where $B \in \mathscr{L}_2$. There exists $C \in \mathscr{L}_2$, $C \subseteq B'$, such that $\gamma(C) = \gamma(B') = 1$. Since $C' \cup B' \supseteq B \cup B' = Y$, then by the argument in the above paragraph, there is $A_k \in \mathscr{L}_1$ such that $\mu(A_k') = 1$ or $\mu \in W(A_k)'$.

Consider any $\lambda \in W(A_k)'$, with $\tilde{T}\lambda = \rho$. Let D be in $\mathscr{L}_1$, $D \subseteq A_k'$, and $\lambda(D) = \lambda(A_k') = 1$. There is $A \in \mathscr{L}_1$, $A \subseteq A_k'$, such that $\mu(A) = 1$. Either $\overline{T(A)} \subseteq C'$ or $\overline{T(A)} \subseteq B'$. If $\overline{T(A)} \subseteq C'$, then $\gamma(\overline{T(A)}) = 1$ and $\gamma(C') = 1$: contradiction. Hence $\overline{T(A)} \subseteq B'$. The WO property then implies that $\overline{T(D)} \subseteq B'$. Now $\tilde{T}\lambda(\overline{T(D)}) = \rho(\overline{T(D)}) \geq \lambda T^{-1}(\overline{T(D)}) \geq \lambda(D) = 1$, where $\overline{T(D)} \in \mathscr{L}_2$. Therefore $\rho(B') = 1$ or $\rho \in W(B)'$. Hence $\tilde{T}(W(A_k)') \subseteq W(B)'$.

II. $T^{-1}(\mathscr{L}_2)$ *Semi-Separates* $\mathscr{L}_1$ *and* T *is Surjective.*

The lattice $T^{-1}(\mathscr{L}_2)$ *semi-separates* $\mathscr{L}_1$ if for $A \in \mathscr{L}_1$, $B \in \mathscr{L}_2$, $T^{-1}(B) \cap A = \phi$ there exists $B_0 \in \mathscr{L}_2$ such that $A \subseteq T^{-1}(B_0)$ and $T^{-1}(B) \cap T^{-1}(B_0) = \phi$. Cohen [3] proved that in this case, for each $\mathscr{L}_1$-regular measure μ on $\mathscr{A}(\mathscr{L}_1)$, the restriction $\mu|_{\mathscr{A}(T^{-1}(\mathscr{L}_2))}$, which is to be abbreviated as $\mu|_{T^{-1}(\mathscr{L}_2)}$, is $T^{-1}(\mathscr{L}_2)$-regular.

THEOREM 2. *The sets* X *and* Y *have lattices* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *respectively, both satisfying the conditions of the proposition in section* 2, *while* $\mathscr{W}_1$ *and* $\mathscr{W}_2$ *are bases for the closed sets in* $I_R(\mathscr{L}_1)$ *and* $I_R(\mathscr{L}_2)$ *respectively. If* $T : X \to Y$ *is* $\mathscr{L}_1$-$\mathscr{L}_2$ *continuous and surjective, and* $T^{-1}(\mathscr{L}_2)$ *semi-separates* $\mathscr{L}_1$, *then* T *can be extended to a surjective* $\mathscr{W}_1$-$\mathscr{W}_2$ *continuous map* $\tilde{T} : I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$–*which is therefore continuous with respect to the Wallman topologies* O_{w_1} *in* $I_R(\mathscr{L}_1)$ *and* O_{w_2} *in* $I_R(\mathscr{L}_2)$–*with* $\tilde{T}\mu = \gamma = \mu T^{-1}$.

Proof. 1. The image $\gamma = \mu T^{-1} \in I_R(\mathscr{L}_2)$, since $\mu|_{T^{-1}(\mathscr{L}_2)}$ is $T^{-1}(\mathscr{L}_2)$-regular and T is surjective.

2) The map $\tilde{T}$ is $\mathscr{W}_1 - \mathscr{W}_2$ continuous with $T^{-1} W(B) = W(T^{-1}(B))$, for any $B \in \mathscr{L}_2$.

3) The map $\tilde{T}$ is surjective: Suppose $\gamma \in I_R(\mathscr{L}_2)$. Define λ on $\mathscr{A}(T^{-1}(\mathscr{L}_2)) = T^{-1}(\mathscr{A}(\mathscr{L}_2))$ by $\lambda(T^{-1}(B)) = \gamma(B)$, for any $B \in \mathscr{A}(\mathscr{L}_2)$. If

$\lambda(T^{-1}(B)) = 1$, there is $L_2 \in \mathcal{L}_2$ with $L_2 \subseteq B$ and $\gamma(L_2) = 1$. Since $T^{-1}(B) \supseteq T^{-1}(L_2)$, $\lambda(T^{-1}(L_2)) = \gamma(L_2) = 1$. Extend λ to $\mu \in I_R(\mathcal{L}_1)$ [3]. For any $B \in \mathcal{A}(\mathcal{L}_2) \cdot \tilde{T}\mu(B) = \mu T^{-1}(B) = \lambda T^{-1}(B) = \gamma(B)$.

4) As in Theorem 1, $\tilde{T}$ extends T.

III. *T is $\mathcal{L}_1$–$\mathcal{L}_2$ Closed and Surjective.*

THEOREM 3. In the setting of Theorem 2, if $T: X \to Y$ is $\mathcal{L}_1$–$\mathcal{L}_2$ continuous, surjective, and $\mathcal{L}_1$–$\mathcal{L}_2$ closed, then: 1) $T^{-1}(\mathcal{L}_2)$ semi-separates $\mathcal{L}$, 2) T can be extended to a $\mathcal{W}_1$–$\mathcal{W}_2$ continuous (therefore continuous with respect to the Wallman topologies), surjective, and $\mathcal{W}_1$–$\mathcal{W}_2$ closed map $\tilde{T}: I_R(\mathcal{L}_1) \to I_R(\mathcal{L}_2)$ defined as $\tilde{T}\mu = \mu T^{-1}$; and 3) $\tilde{T}$ is also closed with respect to the Wallman topologies O_{w_1} in $I_R(\mathcal{L}_1)$ and O_{w_2} in $I_R(\mathcal{L}_2)$.

Proof. 1) Suppose for $B \in \mathcal{L}_2$, $A \in \mathcal{L}_1$, $T^{-1}(B) \cap A = \phi$, then we claim that $B \cap TA = \phi$: If $y \in B \cap TA$, then $y = Tx$ for some $x \in A$, and $x \in T^{-1} y \cap A \subseteq T^{-1} B \cap A$, which is a contradiction. As T is $\mathcal{L}_1$–$\mathcal{L}_2$ closed, $TA = B_0 \in \mathcal{L}_2$ and $B \cap B_0 = \phi$. Hence $T^{-1}(B_0) = T^{-1}T(A) \supseteq A$, and $T^{-1}(\mathcal{L}_2)$ semi-separates $\mathcal{L}_1$.

2) It follows from Theorem 2 that the extension $\tilde{T}$ exists and is $\mathcal{W}_1$–$\mathcal{W}_2$ continuous and surjective. To show that $\tilde{T}$ is $\mathcal{W}_1$–$\mathcal{W}_2$ closed, first prove that $(\tilde{T})^{-1}\gamma = \bigcap_{\substack{B \in \mathcal{L}_2 \\ \gamma(B)=1}} W(T^{-1}(B))$: Let $\mu \in (\tilde{T})^{-1}\gamma$, then $\gamma \in \tilde{T}\mu = \mu T^{-1}$. For $B \in \mathcal{L}_2$ with $\gamma(B) = 1$, $\mu T^{-1}(B) = 1$, or $\mu \in W(T^{-1}(B))$. Conversely, if $\mu T^{-1}(B) = 1$ for all $B \in \mathcal{L}_2$ with $\gamma(B) = 1$, then $\gamma \leq \mu T^{-1}$ on $\mathcal{L}_2$. Hence $\gamma = \mu T^{-1} = \tilde{T}\mu$, and $\mu \in (\tilde{T})^{-1}$. Now show that $W(TA) = \tilde{T}(W(A))$: Let $\gamma \in W(TA)$. Since $T^{-1}\gamma = \bigcap_{\substack{B \in \mathcal{L}_2 \\ \gamma(B)=1}} W(T^{-1}(B))$, consider $\bigcap_{\substack{B \in \mathcal{L}_2 \\ \gamma(B)=1}} W(T^{-1}(B)) \cap W(A)$. If the intersection is empty, then for some B_0 with $\gamma(B_0) = 1$, $W(T^{-1}B_0) \cap \cap W(A) = \phi$: Otherwise, the family $\{W(T^{-1}B) \cap W(A)\}$ has the finite intersection property, and as $I_R(\mathcal{L}_1)$ is compact, the intersection would not be empty. Now $W(T^{-1}B_0) \cap W(A) = W(T^{-1}B_0 \cap A) = \phi$ implies that $T^{-1}B_0 \cap A = \phi$, or $A \subseteq (T^{-1}B_0)' = T^{-1}(B_0')$, and that $TA \subseteq B_0'$. We also have $\gamma(B_0') = 0$, so $\gamma(TA) = 0$, but $\gamma \in W(TA)$. Let μ be in the non-empty intersection, so $\mu \in (\tilde{T})^{-1}\gamma$, where $\mu(A) = 1$. Hence $\gamma \in \tilde{T}(W(A))$. The reverse inclusion follows, because if $\gamma = \tilde{T}\mu = \mu T^{-1}$ and $\mu(A) = 1$, then $\gamma T(A) = \mu T^{-1} T(A) \geq \mu(A) = 1$.

3) The map $\tilde{T}$ is closed with respect to the Wallman topologies; in fact, for any closed set $F = \cap W(A_\alpha)$ in $I_R(\mathcal{L}_1)$, $\tilde{T}(F) = \tilde{T}(\cap W(A_\alpha)) = \cap \tilde{T}(W(A_\alpha)) = \cap W(TA_\alpha)$. The last equality follows from part 2. In general, $\tilde{T}(\cap W(A_\alpha)) \subseteq \cap \tilde{T}(W(A_\alpha))$. Let $\gamma \in \cap \tilde{T}(W(A_\alpha))$ and $D = (\tilde{T})^{-1}\gamma$. Note that $\{\gamma\}$ is closed in the T_1 space $I_R(\mathcal{L}_2)$, so D is closed in $I_R(\mathcal{L}_1)$. Claim that $D \cap W(A_\alpha) \neq \phi$: Otherwise, $W(A_\alpha) \subseteq D'$ and $\tilde{T}(W(A_\alpha)) \subseteq \tilde{T}(D')$; so $\gamma \in \tilde{T}(D')$. But $\gamma = \tilde{T}\mu$, where $\mu \in D'$, is a contradiction since $\mu \in (\tilde{T})^{-1}\gamma = D$.

Also, $(D \cap W(A_\alpha)) \cap (D \cap W(A_\beta)) = D \cap W(A_\alpha \cap A_\beta) \neq \phi$. Hence $\{D \cap W(A_\alpha)\}$, where $D \cap W(A_\alpha)$ is closed in the compact space $I_R(\mathscr{L}_1)$, has the finite intersection property. There is $T \in \cap (D \cap W(A_\alpha)) = D \cap \cap (W(A_\alpha))$, and $\tilde{T}\tau = \gamma$. Hence $\gamma \in \tilde{T}(\cap W(A_\alpha))$.

We note that if $\mathscr{L}_2$ is normal and $\tilde{T}: I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$ is continuous then $\tilde{T}$, as a continuous map from a compact space to a Hausdorff space, is naturally closed.

APPLICATIONS. We give a few examples to illustrate some of the many possible applications:

1) In the case where X and Y are zero-dimensional Hausdorff spacees with $\mathscr{L}_1$ and $\mathscr{L}_2$ as the sets of all clopen sets in X and Y respectively, $T: X \to Y$ in $\mathscr{L}_1$-$\mathscr{L}_2$ continuous if and only if T is continuous topologically. Also, $T^{-1}(\mathscr{L}_2)$ semi-separates $\mathscr{L}_1$, as $\mathscr{L}_2$ is closed under complement.

An $\mathscr{L}_1$-$\mathscr{L}_2$ continuous map $T: X \to Y$ can be, by Theorem 2, extended to a $\mathscr{W}_1$-$\mathscr{W}_2$ continuous (therefore continuous) $\tilde{T}: I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$ on the Banaschewski compactification. Since every measure on $\mathscr{A}(\mathscr{L}_2) = \mathscr{L}_2$ is automatically $\mathscr{L}_2$-regular, we need not assume that T is surjective in this case. This generalizes a well-known theorem of the Banaschewski compactification [10]: If f is a continuous map of a Hausdorff zero-dimensional space X into a compact zero-dimensional space Y, then (viewing X as a subspace of $\beta_0 X$ and identifuing Y with $\beta_0 Y$) f has a unique continuous extension $\tilde{f}: \beta_0 X \to Y$.

2) If X and Y are T_1 spaces with $\mathscr{L}_1 = \mathscr{F}_1$ and $\mathscr{L}_2 = \mathscr{F}_2$, the sets of all closed sets in X and Y respectively, then $\mathscr{L}_1$-$\mathscr{L}_2$ continuity is equivalent to topological continuity. By Theorem 3, an $\mathscr{L}_1$-$\mathscr{L}_2$ continuous, surjective, and $\mathscr{L}_1$-$\mathscr{L}_2$ closed (therefore closed) map $T: X \to Y$ can be extended to a $\mathscr{W}_1$-$\mathscr{W}_2$ continuous (therefore continuous), surjective, $\mathscr{W}_1$-$\mathscr{W}_2$ closed, and closed $\tilde{T}: I_R(\mathscr{L}_1) \to I_R(\mathscr{L}_2)$ on the Wallman compactification. Theorem 3 is a generalization of a theorem by Arhangel'skii [8]: Any closed continuous mapping f of a space X onto a space Y can be extended to a closed continuous map of ωX onto ωY.

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