Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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The behaviour of solutions of a class of nonlinear integral equations of the Volterra type

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.1, p. 9–16.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_1_9_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Equazioni funzionali. — *The behaviour of solutions of a class of nonlinear integral equations of the Volterra type.* Nota di NICOLAIE LUCA, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Con l'uso di una condizione di positività del metodo delle frequenze, si stabiliscono condizioni sufficienti affinché tutte le soluzioni di un'equazione integrale non lineare del tipo di Volterra risultino a quadrato integrabile.

1. INTRODUCTION

The problem of the behaviour of solutions for integral equations of Volterra type has been studied by many Authors. The results were obtained by systematic application of the frequency method, fundamented for differential systems by V.M. Popov [9–12] and then extended for integral and integro-differential equations by C. Corduneanu [2–7]. Another method for the study of the functional equations was applied by J. Kudrewicz [6]. This method uses essentially a positivity condition.

Using the frequency method, V. Barbu [1] established some theorems concerning existence and behaviour of the solutions of Volterra's type non-linear scalar integral equations. Later, A. Halanay [5], applying systematically the previous methods, established sufficient conditions under which the solutions of a nonlinear integral equation of the Volterra type are square integrable on the half-axis.

In this paper, using devices analogous to those in Halanay's paper [5] and in the Author's one [8], sufficient conditions are established so that the solutions of Volterra's type nonlinear vector integral equation should be square integrable on the half-axis.

2. NOTATIONS AND DEFINITIONS

To formulate the results, we shall first mention the significance of the notations that will be used in the following, namely we shall denote by:

R – the real line, $R^2 = R \times R$, $R^+ = \{t \mid t \in R, t \geq 0\}$;

C – the set of complex numbers; $\bar{z}$ is the conjugate of $z \in C$; $E^n = \{x \mid x = (x_1, \cdots, x_n), x_i \in R, i = \overline{1, n}\}$, and if $x, y \in E^n$, then

$$(x, y) = \sum_{i=1}^{n} x_i y_i \quad , \quad |x| = (x, x)^{1/2};$$

(*) Nella seduta dell'8 gennaio 1977.

E^{n^2} – the set of nth-order matrices, whose elements are real or complex numbers, and if $A \in E^{n^2}$, then $A^* = \overline{(A')}$;

$$\Delta = \{(t, \tau) \mid (t, \tau) \in R^2, \ 0 \leq \tau \leq t\};$$

$$L^2([0, T], E^n) = \left\{x \mid x : [0, T] \to E^n, \|x\|_T = \left(\int_0^T |x(t)|^2 \, dt\right)^{1/2} < +\infty\right\},$$

and if $x, y \in L^2([0, T], E^n)$, then:

$$\langle x, y \rangle_T = \int_0^T (x(t), y(t)) \, dt.$$

If $M \in E^{n^2}$, then:

$$|M|_* = \inf\{\mu \mid \mu \geq 0, |(Mx, x)| \geq \mu |x|^2\}.$$

DEFINITION. We say that $M \in E^{n^2}$ if a positive matrix if $(Mx, x) \geq 0$, $\forall x \in E^n$. The matrix M is positive definite if there exists a constant $\eta > 0$ such that $(Mx, x) \geq \eta |x|^2$. A positive matrix M will be denoted by $M \geq 0$, and a positive definite matrix M will be denoted by $M > 0$.

3. We establish in this section the main result of this paper.

Let us consider the functional equation:

$$\sigma(t) = (F\sigma)(t) + \int_0^t \mathscr{K}(t, \tau)(G\sigma)(\tau) \, d\tau, \tag{3.1}$$

where $F\sigma$ and $G\sigma$ are two given operators and $\mathscr{K}(t, \tau)$ is a given matrix-function.

More precisely if $f : R^+ \times E^n \to E^n$ and $g : R^+ \times E^n \to E^n$ are two given functions, then:

$$(F\sigma)(t) = f(t, \sigma(t)) \quad , \quad (G\sigma)(t) = g(t, \sigma(t)). \tag{3.2}$$

We assume that:

$$\mathscr{K} : \Delta \to E^{n^2}.$$

Now, the following basic result can be easily stated.

THEOREM 3.1. *If the following conditions are satisfied:*

a) *$f(t, \sigma)$ and $g(t, \sigma)$ are measurable functions with respect to t on R^+, for any fixed σ in E^n, and continuous with respect to σ in E^n, for any $t \in R^+$;*

b) *there exist two constants* $L_1 \geq 0$, $L_2 > 0$ *and two functions* $\varphi_1, \varphi_2 \in L^2(R^+)$, *such that*:

$$|F\sigma| \leq L_1 |\sigma| + \varphi_1(t),$$
$$|G\sigma| \leq L_2 |\sigma| + \varphi_2(t) \quad , \quad t \in R^+ , \ \sigma \in E^n;$$

c) $\mathcal{K}(t, \tau) \in L^2(\Delta)$;

d) $\exists P, Q, J \in E^{n^2}$, $Q > 0$, $J \geq 0$, *such that*:

(3.4) $$\langle P\mathcal{A}u, u\rangle_T - \langle Q\mathcal{A}u, \mathcal{A}u\rangle_T + \langle Ju, u\rangle_T \geq 0,$$
$$\forall T > 0 \quad , \quad \forall u \in L^2([0, T], E^n),$$

where:

$$(\mathcal{A}u)(t) \triangleq \int_0^t \mathcal{K}(t, \tau) u(\tau) d\tau;$$

e) $L_1, L_2, \varphi_1, \varphi_2, P, Q, J$ *are such that*:

(3.5) $$\lambda - \alpha > 0,$$

where λ *and* α *will be specified, then any solution* σ *of equation* (3.1) *belongs to* $L^2([0, T], E^n)$.

Proof. Taking into account the significance of the operator, equation (3.1) may be written as:

(3.6) $$\sigma(t) = (F\sigma)(t) + (\mathcal{A}G\sigma)(t).$$

Let $\sigma(t)$ be a solution of equation (3.6). From hypotheses *a)*–*c)* there results that $\sigma \in L^2([0, T], E^n)$. Putting in (3.4) $u(t) = (G\sigma)(t)$ and taking into account (3.6), we obtain:

$$\langle P\sigma(t) - P(F\sigma)(t), (G\sigma)(t)\rangle_T - \langle Q\sigma(t) - Q(F\sigma)(t), \sigma(t) - (F\sigma)(t)\rangle_T +$$
$$+ \langle J(G\sigma)(t), (G\sigma)(t)\rangle_T \geq 0, \qquad \forall T > 0,$$

whence:

(3.7) $$\langle Q\sigma(t), \sigma(t)\rangle_T \leq \langle P\sigma(t), (G\sigma)(t)\rangle_T - \langle P(F\sigma)(t), (G\sigma)(t)\rangle_T +$$
$$+ 2\langle Q(F\sigma)(t), \sigma(t)\rangle_T - \langle Q(F\sigma)(t), (F\sigma)(t)\rangle_T +$$
$$+ \langle J(G\sigma)(t), (G\sigma)(t)\rangle_T, \qquad \forall T > 0.$$

But:

(3.8) $$|\langle P\sigma(t), (G\sigma)(t)\rangle_T| \leq |P| \, \|\sigma\|_T (L_2 \|\sigma\|_T + \|\varphi_2\|_T),$$
$$|\langle P(F\sigma)(t), (G\sigma)(t)\rangle_T| \leq |P| \, \|F\sigma\|_T \|G\sigma\|_T \leq$$
$$\leq |P| (L_1 \|\sigma\|_T + \|\varphi_1\|_T)(L_2 \|\sigma\|_T + \|\varphi_2\|_T),$$
$$|\langle Q(F\sigma)(t), \sigma(t)\rangle_T| \leq |Q| \, \|\sigma\|_T (L_1 \|\sigma\|_T \|\varphi_1\|_T),$$
$$|\langle J(G\sigma)(t), (G\sigma)(t)\rangle_T| \leq |J| (L_2 \|\sigma\|_T + \|\varphi_2\|_T)^2.$$

Taking into account the hypotheses *a*)-*d*) and the inequalities (3.8), we obtain from (3.7):

$$\langle Q\sigma(t), \sigma(t)\rangle_T \leq \alpha \|\sigma\|_T^2 + \beta \|\sigma\|_T + \gamma, \tag{3.9}$$

where:

$$\begin{aligned} \alpha &= |P| L_2 (1 + L_1) + 2|Q| L_1 + |J| L_2, \\ \beta &= |P| \|\varphi_2\| (1 + L_1) + \|\varphi_1\| (|P| L_2 + 2|Q|) + 2|J| L_2 \|\varphi_2\|, \\ \gamma &= |P| \|\varphi_1\| \cdot \|\varphi_2\| + |J| \|\varphi_2\|^2, \\ \|\varphi\| &= \left(\int_0^\infty \varphi^2(t)\, dt\right)^{1/2}. \end{aligned} \tag{3.10}$$

Because $Q > 0$ it follows that there exists a $\lambda > 0$, such that:

$$(Qx, x) \geq \lambda |x|^2, \qquad \forall x \in E^n. \tag{3.11}$$

We shall also denote by λ the greatest number λ for which (3.11) is satisfied, i.e. $|Q|_* = \lambda$. Then, from (3.9) and (3.11), we obtain:

$$(\lambda - \alpha) \|\sigma\|_T^2 - \beta \|\sigma\|_T - \gamma \leq 0,$$

whence:

$$0 \leq \|\sigma\|_T \leq \frac{\beta + \sqrt{\beta^2 + \gamma(\lambda - \alpha)}}{\lambda - \alpha} = B, \qquad \forall T > 0. \tag{3.12}$$

Taking into account (3.10) and condition *e*) of the theorem it follows that $B > 0$. As

$$B = B(\alpha, \beta, \gamma, \lambda) = B(L_1, L_2, \|\varphi_1\|, \|\varphi_2\|, |P|, |Q|, |J|)$$

it follows that B is independent of T. Then (3.12) implies $\|\sigma\| \leq B$ and therefore $\sigma \in L^2(R^+, E^n)$.

Remark 3.1. If $f(t, \sigma) \equiv f(t) \in L^2(R^+, E^n)$ then taking $P = pI$ (p and I being a scalar and the unit matrix respectively), $Q = I$, $J = 0$, $L_1 = 0$, $\varphi_1 = f$, the positivity condition (3.4) becomes:

$$p\,\langle \mathcal{A}u, u\rangle_T - \langle \mathcal{A}u, \mathcal{A}u\rangle_T \geq 0, \qquad \forall T > 0, \quad u \in L^2([0, T], E^n).$$

In this case $\lambda = 1$, $\alpha = |p| L_2$, $\beta = |p| \|\varphi_2\|$, $\gamma = 0$ and condition (3.5) takes the form:

$$1 - pL_2 > 0.$$

Problems of this type were considered by A. Halanay [5] (for $n \geq 1$) and by V. Barbu [1] (for $n = 1$).

Remark 3.2. Equations of the form (3.1) occur frequently in the theory of automatic control systems. Some systems of this type are described by equations of the form:

$$\dot{x}(t) = Ax(t) + R(G\sigma)(t), \qquad \sigma(t) = Sx(t) + (\Phi\sigma)(t),$$

where $x = (x_1, \cdots, x_n)$, $\sigma = (\sigma_1, \cdots, \sigma_m)$ are vectors, A, R, S, are constant matrices of type $n \times n$, $m \times n$, $n \times m$ respectively and $G\sigma$ nad $\Phi\sigma$ are two given n and m-vector functions, respectively.

Such a system with the initial values $t = 0$, $x^0 = \theta$ can be reduced to an integral equation of Volterra type. Indeed, by applying the variation of constants formula, we obtain from the first equation:

$$x(t) = e^{At} x^0 + \int_0^t e^{A(t-s)} R(G\sigma)(s)\, ds .$$

If we substitute the above expression for $x(t)$ in the second equation of the considered system we obtain:

$$\sigma(t) = Se^{At} x^0 + \int_0^t Se^{A(t-s)} R(G\sigma)(s)\, ds + (\Phi\sigma)(t),$$

which is an equation of the form (3.1), with:

$$(F\sigma)(t) = (\Phi\sigma)(t) + Se^{At} x^0 \quad , \quad \mathcal{K}(t, \tau) = Se^{A(t-\tau)} R .$$

In the following section we shall prove that for a certain class of kernels $\mathcal{K}(t, \tau)$, very interesting in applications, condition (3.4) is satisfied if a certain frequential condition is satisfied.

4. If u belongs to $L^1(R^+, E^n)$ or $L^2(R^+, E^n)$ and $\mathcal{H}$ belongs to $L^1(R^+, E^{n^2})$ or $L^2(R^+, E^{n^2})$, we denote by $\tilde{u}(i\omega)$ and $\tilde{\mathcal{H}}(i\omega)$ its Laplace transform, i.e.:

$$\tilde{u}(i\omega) = \int_{-\infty}^{\infty} u(t)\, e^{i\omega t}\, dt \quad , \quad \tilde{\mathcal{H}}(i\omega) = \int_{-\infty}^{\infty} \mathcal{H}(t)\, e^{i\omega t}\, dt , \qquad \omega \in R$$

where $u(t) = \theta$, $\mathcal{H}(t) = 0$ for $t < 0$. We point out that θ is the null vector, O is the null matrix and the Laplace transform of a matrix is the matrix whose elements are the Laplace transforms of the corresponding elements of the given matrix.

THEOREM 4.1. *If the following conditions are satisfied*:

a_1) $\mathcal{K}(t, \tau) = \mathcal{K}(t - \tau)$, $\mathcal{K}(s) \in L^1(R^+, E^{n^2}) \cap L^2(R^+, E^{n^2})$;

b_1) *there exist three matrices* P, Q, J, $Q > 0$, $J \geq 0$, *such that*:

$$\text{(4.1)} \qquad P\tilde{\mathcal{K}}(i\omega) + \tilde{\mathcal{K}}^*(i\omega) P' - 2Q\tilde{\mathcal{K}}(i\omega)\tilde{\mathcal{K}}^*(i\omega) + 2J \geq 0 , \qquad \forall \omega \in R,$$

where:

$$\tilde{\mathcal{K}}^*(i\omega) = \overline{((\tilde{\mathcal{K}}(i\omega))')},$$

then condition (3.4) *is satisfied.*

Proof. From hypothesis a_1) there results that if $u \in L^2([0, T], E^n)$, then $\mathscr{A}u \in L^2([0, T], E^n)$. Relation (3.4) may also be written as:

$$\int_0^T (P(\mathscr{A}u)(t), u(t))\, dt - \int_0^T (Q(\mathscr{A}u)(t), (\mathscr{A}u)(t))\, dt + \int_0^T (Ju(t), u(t))\, dt \geq 0, \qquad \forall T > 0. \tag{4.2}$$

If $u \in L^2([0, T], E^n)$, we define:

$$u_T(t) \triangleq \begin{cases} u(t), & \text{for } 0 \leq t \leq T, \\ 0, & \text{for } t > T, \end{cases} \tag{4.3}$$

$$v_T(t) \triangleq \begin{cases} \int_0^t \mathscr{K}(t-\tau)\, u(\tau)\, d\tau, & \text{for } 0 \leq t \leq T, \\ \int_0^T \mathscr{K}(t-\tau)\, u(\tau)\, d\tau, & \text{for } t > T. \end{cases} \tag{4.4}$$

From (4.3) and (4.4) it follows that:

$$v_T(t) = \int_0^t \mathscr{K}(t-\tau)\, u_T(\tau)\, d\tau. \tag{4.5}$$

Taking into account (4.3) and (4.5), relation (4.2) becomes:

$$\int_0^T (Pv_T(t), u(t))\, dt - \int_0^T (Qv_T(t), v_T(t))\, dt + \int_0^T (Ju(t), u(t))\, dt \geq 0,$$

or:

$$\int_0^\infty [(Pv_T(t), u_T(t)) - (Qv_T(t), v_T(t)) + (Ju_T(t), u_T(t))]\, dt + \int_T^\infty (Qv_T(t), v_T(t))\, dt \geq 0. \tag{4.6}$$

It is obvious that, because $Q > 0$, if the following inequality is satisfied:

$$\int_0^\infty [(Pv_T(t), u_T(t)) - (Qv_T(t), v_T(t)) + (Ju_T(t), u_T(t))]\, dt \geq 0, \tag{4.7}$$

then inequality (4.6) is a fortiori satisfied.

From (4.3), (4.5) and hypothesis a_1) there results that u_T and v_T belong to $L^1(R^+, E^n) \cap L^2(R^+, E^n)$. Then, taking into account Parseval's equality, inequality (4.7) can be written as:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\,[(P\tilde{v}_T(i\omega), \tilde{u}_T(i\omega)) - (Q\tilde{v}_T(i\omega), \tilde{v}_T(i\omega)) + (J\tilde{u}_T(i\omega), \tilde{u}_T(i\omega))]\, d\omega \geq 0, \qquad \forall\omega\in R, \tag{4.8}$$

where:

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i .$$

Taking into account (4.5), we can write:

$$\tilde{v}_T(i\omega) = \tilde{\mathcal{K}}(i\omega)\, \tilde{u}_T(i\omega) . \tag{4.9}$$

Then (4.8) can be written as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\,[(P\tilde{\mathcal{K}}(i\omega)\tilde{u}_T(i\omega), \tilde{u}_T(i\omega)) - (Q\tilde{\mathcal{K}}(i\omega)\tilde{u}_T(i\omega), \tilde{\mathcal{K}}(i\omega)\tilde{u}_T(i\omega)) + (J\tilde{u}_T(i\omega), \tilde{u}_T(i\omega))]\, d\omega \geq 0, \qquad \forall\omega\in R, \tag{4.10}$$

or:

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} ((P\tilde{\mathcal{K}}(i\omega) + \tilde{\mathcal{K}}^*(i\omega)P' - 2Q\tilde{\mathcal{K}}(i\omega)\tilde{\mathcal{K}}^*(i\omega) + 2J)\tilde{u}_T(i\omega), \tilde{u}_T(i\omega))\, d\omega \geq 0, \qquad \forall\omega\in R. \tag{4.11}$$

Taking into account (4.1), it follows that (4.11) is satisfied. q.e.d.

The following theorem is then easily proved:

THEOREM 4.2. *Let us assume that the conditions of Theorem 4.1 and conditions a), b), c) and e) from Theorem 3.1 are satisfied. Then any solution of equation* (3.1) *belongs to* $L^2(R^+, E^n)$.

Indeed, conditions *a*), *b*), *c*), *e*) being satisfied and the conditions of Theorem 4.1 implying condition *d*) of the same theorem, the statement of Theorem 4.2 follows immediately.

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