Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Kähler submanifolds satisfying a certain condition on normal connection

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.1, p. 30–35.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_1_30_0>

Geometria differenziale. — *Kähler submanifolds satisfying a certain condition on normal connection.* Nota di IKUO ISHIHARA, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si stabiliscono (nel § 4) quattro teoremi sulle sottovarietà di Kähler che soddisfano alla condizione (N) qui specificata nel § 2.

§ 1. INTRODUCTION

Smyth [10] has given the classification of complex hypersurfaces of the simply-connected complex space forms which are Einstein manifolds. Chen and Lue [2] have studied Kähler submanifolds in a Kähler manifold, when their normal bundle is flat, and obtained interesting results.

In this paper, we shall study Kähler submanifolds with a condition (N) stated in § 2 in a Kähler manifold or in a complex space form and prove four theorems.

§ 2. PRELIMINARIES

Let M^n be a complex n-dimensional Kähler manifold with complex structure J and metric tensor g. We denote the covariant differentiation in M^n by ∇. Then the curvature tensor R of M^n is given by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ for any tangent vector fields X and Y. Then, as is well known, the curvature tensor R of M^n satisfies the following formulas

$$R(JX, JY) = R(X, Y) \quad , \quad R(X, Y) J = JR(X, Y) , \tag{2.1}$$

$$R(X, Y) Z + R(Y, Z) X + R(Z, X) Y = 0 , \tag{2.2}$$

$$R(X, Y; Z, W) = R(Z, W; X, Y) = -R(Y, X; Z, W) = -R(X, Y; W, Z) , \tag{2.3}$$

for any tangent vector fields X, Y, Z and W, where $R(X, Y; Z, W) = g(R(X, Y) Z, W)$. Let M^n be isometrically immersed in a Kähler manifold $\bar{M}^{n+p}$ of complex dimension $n+p$ as a complex submanifold. If $\bar{J}, \bar{g}, \bar{\nabla}$ and $\bar{R}$ denote the complex structure, the metric tensor, the covariant differentiation and the curvature tensor of $\bar{M}^{n+p}$, respectively, then for any tangent vector fields X, Y and any normal vector field N on M^n, the Gauss-Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \quad , \quad \bar{\nabla}_X N = -A_N(X) + D_X N ,$$

(*) Nella seduta dell'8 gennaio 1977.

where $\bar{g}(B(X, Y), N) = g(A_N(X), Y)$ and D is the linear connection induced in the normal bundle $T(M^n)^{\perp}$. A and B are both called the second fundamental form of M^n. Let $R^{\perp}$ be the curvature tensor associated with D in $T(M^n)^{\perp}$, i.e., $R^{\perp}(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$. Then the equations of Gauss and Ricci are given respectively by

$$(2.4) \qquad \bar{R}(X, Y; Z, W) = R(X, Y; Z, W) + \bar{g}(B(X, Z), B(Y, W)) - \bar{g}(B(Y, Z), B(X, W)),$$

$$(2.5) \qquad \bar{R}(X, Y; N, N') = R^{\perp}(X, Y; N, N') - g([A_N, A_{N'}](X), Y),$$

where X, Y, Z, W are arbitrary vector fields tangent to M^n and N, N' are arbitrary vector fields normal to M^n. Moreover, we have

$$(2.6) \qquad A_{\bar{J}N} = JA_N \quad \text{and} \quad JA_N = -A_N J.$$

Thus we have Trace B = 0, which means that M^n is a minimal submanifold in $\bar{M}^{n+p}$.

In the present paper, we shall study a submanifold M^n in $\bar{M}^{n+p}$ satisfying the condition

$$(N) \qquad R^{\perp}(X, Y) = \rho g(X, JY)\bar{J},$$

where X and Y are arbitrary vector fields tangent to M^n and ρ is a function on M^n.

§ 3. MODEL SUBMANIFOLDS IN A COMPLEX SPACE FORM

In this section, we shall give some typical model submanifolds in a complex space form for later use. Let M be an n-dimensional Riemannian manifold and take $E_1, \cdots, E_n$ which is a local orthonormal frame on M. Then the Ricci tensor S and the scalar curvature r are given respectively by

$$S(X, Y) = \sum_{i=1}^{n} R(E_i, X; Y, E_i) \quad , \quad r = \sum_{i=1}^{n} S(E_i, E_i),$$

where X and Y are any tangent vector fields on M. An n-dimensional Riemannian manifold M with Riemannian metric g is called an Einstein space if its Ricci tensor S satisfies the condition $S = rg/n$, where r is a constant. We call the factor r/n the Ricci curvature of the Einstein space.

A Kähler manifold $\bar{M}^{n+p}$ is called a complex space form of holomorphic sectional curvature c if the curvature tensor $\bar{R}$ satisfies

$$(3.1) \qquad \bar{R}(X, Y)Z = \frac{c}{4}(\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{J}Y, Z)\bar{J}X - \bar{g}(\bar{J}X, Z)\bar{J}Y + 2\bar{g}(X, \bar{J}Y)\bar{J}Z).$$

In the following, we denote such a $\bar{M}^{n+p}$ by $\bar{M}^{n+p}(c)$. Then, as is well known, $\bar{M}^{n+p}(c)$ is an Einstein space with Ricci curvature $r/(n+p) = (n+p+1)\,c/2$.

We now introduce some special kinds of Kähler manifolds which will be usefull in the present paper. Let C^{n+2} denote complex Euclidean space of dimension $n+2$ with the natural complex coordinate system $z^0, z^1, \cdots, z^{n+1}$. $P^{n+1}(C)$ will denote complex the $(n+1)$-dimensional projective space, $P^{n+1}(C)$ is a complex analytic manifold which, when endowed with the Fubini-Study metric, is a Kähler manifold of constant holomorphic sectional curvature 1. Then there is a natural holomorphic mapping $\pi : C^{n+2} - \{0\} \rightarrow P^{n+1}(C)$. The submanifold in $P^{n+1}(C)$ determined by $z^{n+1} = 0$ is merely $P^n(C)$, the induced metric being the Fubini-Study metric of $P^n(C)$. The submanifold Q^n in $P^{n+1}(C)$ determined by $(z^0)^2 + \cdots + (z^{n+1})^2 = 0$ is called the n-dimensional complex quadric: Q^n is a compact Kähler submanifold with the metric and complex structure induced from $P^{n+1}(C)$. Moreover Q^n is a compact Einstein manifold if $n \geqq 2$.

D^{n+1} will denote the open unit ball in C^{n+1} endowed with the natural complex structure and the Bergman metric. This is then a Kähler manifold of constant holomorphic sectional curvature -1. The submanifold of D^{n+1} determined by $z^n = 0$ is merely D^n, the induced metric being Bergman metric of D^n.

The complex $(n+1)$-dimensional Euclidean space C^{n+1} endowed with the usual Hermitian metric is a flat Kähler manifold.

In the absence of any statement regarding metrics, each complex manifold introduced above is understood to have the Kähler metric we assigned to it above. The manifolds $P^n(C)$, C^n and D^n are simply-connected complete Kähler manifolds. Moreover, any simply-connected complex space form M of complex dimension n is (after multiplication of the metric of M by a suitable positive constant) holomorphically isometric to $P^n(C)$, C^n or D^n, according as M is of positive, zero or negative holomorphic sectional curvature (see Hawley [3] or Igusa [4]).

§ 4. Theorems

We define

$$(\tilde{\nabla}_X R^{\perp})(Y, Z) N = D_X (R^{\perp}(Y, Z) N) - R^{\perp}(\nabla_X Y, Z) N - R^{\perp}(Y, \nabla_X Z) N - R^{\perp}(Y, Z) D_X N,$$

for any vector fields X, Y and Z tangent to M^n and any vector field N normal to M^n. The $\tilde{\nabla} R^{\perp}$ is called the van der Waerden-Bortolotti covariant derivative of $R^{\perp}$. Then we can easily prove

$$(\tilde{\nabla}_X R^{\perp})(Y, Z) + (\tilde{\nabla}_Y R^{\perp})(Z, X) + (\tilde{\nabla}_Z R^{\perp})(X, Y) = 0,$$

which is called the second Bianchi identity.

LEMMA 4.1. *Let* M^n $(n \geqq 2)$ *be a Kähler submanifold of a Kähler manifold* $\bar{M}^{n+p}$ *and assume that* M^n *satisfies the condition* (N). *Then* ρ *is constant on* M^n.

Proof. From the condition (N), we have

$$(4.1) \qquad (\tilde{\nabla}_X R^{\perp})(Y, Z) = (X\rho) g(Y, JZ) \bar{J} .$$

By using the second Bianchi identity, we obtain

$$(4.2) \qquad (X\rho) g(Y, JZ) + (Y\rho) g(Z, JX) + (Z\rho) g(X, JY) = 0 .$$

If we substitute $Z = JY$ in (4.2), then we have

$$-(X\rho) g(Y, Y) + (Y\rho) g(Y, X) + (JY\rho) g(X, JY) = 0 .$$

Then, assuming that X is perpendicular to Y and JY, we find

$$(X\rho) g(Y, Y) = 0 ,$$

which implies that ρ is constant on M^n. This completes the proof.

THEOREM 4.2. *Let* M^n *be a Kähler submanifold of a Kähler manifold* $\bar{M}^{n+p}$ *and assume that* M^n *satisfies the condition* (N). *Then the Ricci tensors* S *and* $\bar{S}$ *of* M^n *and* $\bar{M}^{n+p}$ *satisfy the following relation*:

$$S(X, Y) = \bar{S}(X, Y) - p\rho g(X, Y)$$

for any vector fields X *and* Y *tangent to* M^n.

Proof. From the definition of Ricci tensors and the equation of Gauss (2.4), we have

$$(4.3) \qquad S(X, Y) = \bar{S}(X, Y) - \sum_{\alpha=1}^{p} (\bar{R}(N^{\alpha}, X; Y, N_{\alpha}) + \bar{R}(\bar{J}N_{\alpha}, X; Y, \bar{J}N_{\alpha})) - \sum_{i=1}^{2n} \bar{g}(B(E_i, X), B(E_i, Y)) ,$$

where $E_1, \cdots, E_{2n}$ are local orthonormal vector fields tangent to M^n and $N_1, \cdots, N_p, \bar{J}N_1, \cdots, \bar{J}N_p$ are local orthonormal vector fields normal to M^n. On the other hand, from the equation of Ricci (2.5) and the condition (N), we find

$$(4.4) \qquad \bar{R}(X, Y; N_{\alpha}, \bar{J}N_{\alpha}) = \rho g(X, JY) - g([A_{N_{\alpha}}, A_{\bar{J}N_{\alpha}}](X), Y) .$$

Hence, by using (2.6), we obtain

$$(4.5) \qquad \bar{R}(X, Y; N_{\alpha}, \bar{J}N_{\alpha}) = \rho g(X, JY) + 2 g(JA^2_{N_{\alpha}}(X), Y) .$$

By (2.2) and (2.3), we get

(4.6) $\bar{R}(X, JY; N_\alpha, \bar{J}N_\alpha) = \bar{R}(N_\alpha, JY; X, \bar{J}N_\alpha) - \bar{R}(N_\alpha, X; JY, \bar{J}N_\alpha)$.

Hence using (2.1) and (2.3) gives

(4.7) $\bar{R}(X, JY; N_\alpha, \bar{J}N_\alpha) = -(\bar{R}(\bar{J}N_\alpha, X; Y, \bar{J}N_\alpha) + \bar{R}(N_\alpha, X; Y, N_\alpha))$.

On the other hand, from (2.6), we find

(4.8) $$\sum_{i=1}^{2n} \bar{g}(B(E_i, X), B(E_i, Y)) = 2\sum_{\alpha=1}^{p} g(A^2_{N_\alpha}(X), Y).$$

Combining (4.3), (4.5), (4.7) and (4.8), we find

$$S(X, Y) = \bar{S}(X, Y) - p\rho g(X, Y)$$

for any vector fields X and Y tangent to M^n. This completes the proof.

From Theorem 4.2, we have immediately the following

THEOREM 4.3. *Let* M^n $(n \geqq 2)$ *be a Kähler submanifold in an Einstein Kähler manifold* $\bar{M}^{n+p}$ *and assume that* M^n *satisfies the condition* (N). *Then* M^n *is also Einstein.*

THEOREM 4.4. *Let* M^n $(n \geqq 2)$ *be a Kähler submanifold in a complex space form* $\bar{M}^{n+p}(c)$ *and assume that* M^n *satisfies the condition* (N). *Then either* M^n *is totally geodesic or* M^n *is an Einstein Kähler hypersurface of* $\bar{M}^{n+p}(c)$ *with scalar curvature* $n^2 c$. *The latter case occurs only when* $c > 0$.

Proof. From (2.5), (3.1) and the condition (N), we have

(4.9) $$\frac{c}{2} g(X, JY)\bar{g}(\bar{J}N, N') = \rho g(X, JY)\bar{g}(\bar{J}N, N') - g([A_n, A_{n'}](X), Y).$$

Substituting $N' = \bar{J}N$ in (4.9) and using (2.6), we have

$$\frac{c}{2} g(X, JY) g(N, N) = \rho g(X, JY) g(N, N) + 2g(JA^2_N(X), Y),$$

or equivalently

(4.10) $$g(A^2_N(X), JY) = \frac{2\rho - c}{4}\bar{g}(N, N) g(X, JY).$$

The equation (4.10) implies

(4.11) $$A^2_N = \frac{2\rho - c}{4}\bar{g}(N, N) I,$$

where I is the identity transformation. Assuming $2\rho = c$, we find $A^2_N = 0$ which means $A_N = 0$, i.e., that M^n is totally geodesic. Next if we assume that

$2\rho \neq c$ and $p \geqq 2$, then using (4.11) and $A^2_{N_\alpha+N_\beta} = A^2_{N_\alpha} + A_{N_\alpha} A_{N_\beta} + A_{N_\beta} A_{N_\alpha} + A^2_{N_\beta}$, we obtain

$$A_{N_\alpha} A_{N_\beta} + A_{N_\beta} A_{N_\alpha} = 0 \qquad \text{for} \quad \alpha \neq \beta . \tag{4.12}$$

On the other hand, from (4.9) we get

$$A_{N_\alpha} A_{N_\beta} - A_{N_\beta} A_{N_\alpha} = 0 . \tag{4.13}$$

The equations (4.12) and (4.13) imply $A_{N_\alpha} A_{N_\beta} = 0$ for $\alpha \neq \beta$ which means $A_{N_\alpha} = 0$ for all α, i.e., that M^n is totally geodesic. When M^n is a Kähler hypersurface, as is well known the non totally geodesic case of M^n occurs only in a complex space form of positive constant holomorphic sectional curvature (see Takahashi [11]). This complets the proof.

From Theorem 4.4, we have immediately the following (see Smyth [10]).

THEOREM 4.5. *If* $n \geqq 2$, *then*

(i) $P^n(C)$ *and the complex quadric* Q^{n+p-1} *are the only complete Kähler submanifolds of* $P^{n+p}(C)$ *which satisfy the condition* (N);

(ii) D^n (*resp.* C^n) *is the only simply-connected complete Kähler submanifold of* D^{n+p} (resp. C^{n+p}) *which satisfies the condition* (N).

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