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On the condition for the existence of a solution of the modified Molodensky's problem in gravity space

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Geodesia. — *On the condition for the existence of a solution of the modified Molodensky's problem in gravity space.* Nota di FERNANDO SANSÒ, presentata (*) dal Socio L. SOLAINI.

RIASSUNTO. — La soluzione del problema di Molodensky nello spazio della gravità porta a porre, in modo indiretto, una condizione sui dati al contorno di tale problema. Nel presente lavoro si modifica la definizione di tale problema introducendo un vettore costante arbitrario nei dati al contorno: si dimostra poi un teorema di incondizionata solubilità del problema così posto, per dati sufficientemente vicini a quelli caratteristici di una sfera omogenea.

Note. For the symbols an the results used in this paper we refer the reader to [1]. *The boundary value problem of physical Geodesy in gravity space.* (F. SANSÒ «Memorie Accademia dei Lincei», 1976) and to [2]. *Discussion on the existence and uniqueness of the solution of Molodensky's problem in gravity space* (F. SANSÒ, «Rendiconti Accademia dei Lincei»).

1. We recall that Molodensky's problem has been shown to be equivalent to the non linear Dirichlet's problem

$$\left\{\begin{array}{lll} \Delta v = \mu \mathrm{F}[v] & \gamma \in \mathrm{D} & \text{(domain containing } \gamma = 0) \\ v = v(\alpha) & \gamma \in \Gamma \equiv \{\gamma(\alpha)\} & \end{array}\right. \tag{1.1}$$

where

$$\mathrm{F}[v] = \mathrm{B}(v, v) \tag{1.2}$$

$$\mathrm{B}(u, v) \quad \text{bilinear operator on } u \text{ and } v$$

$$\|\mathrm{B}(u, v)\|_{\lambda} \leq h \|u\|_{2+\lambda} \|v\|_{2+\lambda}. \tag{1.3}$$

It has been proved that if the condition

$$4 \mu c h \|v(\alpha)\|_{2+\lambda} \leq 1 \quad (c = \text{const. of Schauder's estimate}) \tag{1.4}$$

is satisfied, equation (1.1) has a solution v which is the unique one physically acceptable, on condition that

$$\nabla v(0) = 0. \tag{1.5}$$

We have to notice that

a) if (1.5) is not satisfied, even if equation (1.1) has a solution, Molodensky's problem is not solvable;

b) the relation (1,5) is indeed a condition on the data, that is $\gamma(\alpha)$ and $u(\alpha)$, but a condition in an implicit form;

(*) Nella seduta dell'11 dicembre 1976.

c) due to the statistical nature of $\gamma(\alpha)$ and $u(\beta)$, which are affected by measure and estimation errors, we cannot expect that they are in the class of the acceptable data.

To overcome this difficulty we have two ways: to approach the problem from the statistical point of view, looking for the most probable $u(\alpha), \gamma(\alpha)$ in the class of the acceptable data or to give a slightly modified formulation of Molodensky's problem in order to make it solvable for any data, at most satisfying a condition of the kind of (1.4); this will be done according to the suggestion of L. HÖRMANDER ([3]. *The boundary problems of physical geodesy*, Report 9, Institut Mittag-Leffler, 1975). The idea is to search for a function $v \in C_{2+\lambda}$ and a constant vector $\boldsymbol{c}$ satisfying the system

$$(1,6) \qquad \begin{cases} \Delta v = \mu F[v] & \gamma \in D \\ v = v(\alpha) + \boldsymbol{c} \cdot \gamma(\alpha) & \gamma \in \Gamma \\ \nabla v(0) = 0 . \end{cases}$$

Introducing the Green's operator G of the Laplacian with homogeneous boundary conditions, the harmonic function v_0 with boundary values $v(\alpha)$, and using the notation

$$\nabla v(0) = \nabla_0 v$$
$$GF[v] = Q(v),$$

we can write instead of (1.6) the equivalent system

$$(1.7) \qquad \begin{cases} v - v_0 - \boldsymbol{c} \cdot \gamma - \mu Q(v) = 0 \\ \nabla_0 v_0 + \boldsymbol{c} + \mu \nabla_0 Q(v) = 0 . \end{cases}$$

Furthermore, we recall that in [2] the following majorizations for the operator $Q(v)$ are proved

$$(1.8) \qquad \| Q(v) \| \leq ch \|v\|_{2+\lambda}^2 \quad , \quad \|Q'(v)\| \leq 2\, ch \|v\|_{2+\lambda} .$$

2. Let us recal the

THEOREM I [1]. *Let us consider the equation*

$$(2.1) \qquad \pi(x) + \mu R(x) = 0$$

where $\pi(x)$ *and* $R(x)$ *are operators defined in a ball*

$$\Omega \{\|x - x_0\| \leq r\}$$

(1) See: VAINBERG M. M., *Variational methods for the study of non linear operators.* Holden Day Inc., 1964.

of a Banach space E_x. *Assuming that*

(2.2) $$\pi(x_0) = 0$$

(2.3) $$\|R(x_0)\| \leq \eta \quad , \quad \|R'(x_0)\| \leq \xi \quad , \quad \|R''(x)\| \leq L \qquad (x \in \Omega)$$

(2.4) $$\|\pi'(x_0)^{-1}\| \leq \beta \quad , \quad \|\pi''(x)\| \leq K \qquad (x \in \Omega)$$

(2.5) $$\frac{\beta^2 \eta (K + L\mu)\mu}{(1 - \xi\beta\mu)^2} \leq \frac{1}{2} \quad (\mu \geq 0),$$

the sequence

(2.6) $$x_{n+1} = x_n - [\pi'(x_0)^{-1}][\pi(x_n) + \mu R(x_n)]$$

is convergent in Ω *to a solution* x^* *of* (1.8).

3. Let us choose as E_x the space with elements

(3.1) $$x = \begin{vmatrix} v \\ \boldsymbol{c} \end{vmatrix}$$

and norm

(3.2) $$\|x\| = \max\{\|v\|_{2+\lambda}, \max|c_i|\}$$

where the c_i's are the components of $\boldsymbol{c}$.

We observe that (1.7) is of the form (2.1) if we take

(3.3) $$\pi(x) = \begin{bmatrix} J - v_0 & -\gamma\cdot \\ 0 & I + \nabla_0 v_0 \end{bmatrix} \begin{bmatrix} v \\ \boldsymbol{c} \end{bmatrix}$$

(3.4) $$R(x) = \begin{bmatrix} -Q(v) \\ \nabla_0 Q(v) \end{bmatrix}$$

where J and I are the identity operators respectively in $C_{2+\lambda}$ and R_3.

Putting

(3.5) $$x_0 = \begin{bmatrix} v_0 - \nabla_0 v_0 \cdot \gamma \\ -\nabla_0 v_0 \end{bmatrix}$$

we see that

(3.6) $$\pi(x_0) = 0 .$$

We now want to estimate the quantities η, ξ, β, L, K for the operators (3.3) and (3.4).

We have

(3.7) $$\pi'(x) = \begin{bmatrix} J & -\gamma\cdot \\ 0 & I \end{bmatrix} \quad , \quad \pi'(x)^{-1} = \begin{bmatrix} J & \gamma\cdot \\ 0 & I \end{bmatrix} \quad , \quad \pi''(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Therefore

$$\|\pi'(x_0)^{-1}x\| = \max\{\|v + \gamma\cdot\mathbf{c}\|_{2+\lambda}, \max_i |c_i|\}$$

but, denoting by ρ the diameter of D.

$$\|v + \gamma\cdot\mathbf{c}\|_{2+\lambda} \leq \|v\|_{2+\lambda} + \max_{\Upsilon} |\gamma\cdot\mathbf{c}| + \max_i |c| \leq$$

$$\leq \|v\|_{2+\lambda} + \max_i |c_i| + \rho |\mathbf{c}| \leq (2 + \sqrt{3}\,\rho)\,\|x\|.$$

We have then

$$\|\pi'(x_0)^{-1}\| \leq 2 + \sqrt{3}\,\rho = \beta. \tag{3.8}$$

Besides, from (3.7)

$$\|\pi''(x)\| = 0 = \mathrm{K}, \qquad \forall x. \tag{3.9}$$

Moreover

$$\|\mathrm{R}(x_0)\| = \max\{\|\mathrm{Q}(v_0)\|_{2+\lambda}, \max_i |\nabla_{0i}\mathrm{Q}(v_0)|\}$$

Since

$$|\nabla_{0i}\mathrm{Q}(v_0)| \leq \|\mathrm{Q}(v_0)\|_{2+\lambda}$$

$$\|\mathrm{Q}(v_0)\|_{2+\lambda} \leq ch\,\|v_0\|^2_{2+\lambda}$$

we find

$$\|\mathrm{R}(x_0)\| \leq ch\,\|v_0\|^2_{2+\lambda} = \eta. \tag{3.10}$$

From the relation

$$\mathrm{R}'(x_0) = \begin{bmatrix} \mathrm{Q}'(v_0) & 0 \\ \nabla_0\mathrm{Q}'(v_0) & 0 \end{bmatrix} \tag{3.11}$$

recalling also (1.10), we derive

$$\|\mathrm{R}'(x_0)\| \leq \|\mathrm{Q}'(v_0)\| \leq 2\,ch\,\|v_0\|_{2+\lambda} = \xi. \tag{3.12}$$

Since as it is easy to prove,

$$\|\mathrm{Q}''(v)\| \leq 2\,ch,$$

differentiating (3.11) we get

$$\|\mathrm{R}''(x)\| \leq \|\mathrm{Q}''(v)\| \leq 2\,ch = \mathrm{L}, \qquad \forall x. \tag{3.13}$$

The condition (2.5), using the estimates (3.8), (3.9), (3.10), (3.11), (3.12), is satisfied if

$$4\,\mu ch\,(2 + \sqrt{3}\,\rho)\,\|v_0\|_{2+\lambda} \leq 1. \tag{3.14}$$

At last, we can observe that, in our case, the sequence of the successive approximations (2.6) can be written

$$(3.15)\qquad \begin{cases} v_{n+1} = v_0 + \mu Q(v_n) - \gamma\cdot\nabla_0 v_0 - \mu\gamma\cdot\nabla_0 Q(v_n) \\ \boldsymbol{c}_{n+1} = -\nabla_0 v_0 - \mu\nabla_0 Q(v_n)\,. \end{cases}$$

We can thus conclude by stating the

THEOREM II. *If the condition*

$$(3.16)\qquad 4\,\mu ch\,(2 + \sqrt{3}\,\rho)\,\|v_0\|_{2+\lambda} \leq 1$$

is satisfied, the sequence

$$(3.17)\qquad v_{n+1} = v_0 + \mu Q(v_n) - \gamma\cdot\nabla_0 v_0 + \mu\gamma\cdot\nabla_0 Q(v_n)$$

starting with the function $v_0 - \gamma\cdot\nabla_0 v_0$ *is convergent in* $C_{2+\lambda}$ *to the solution* v^* *of the modified problem* (1.7), *where the constant vector* c *results to be*

$$(3.18)\qquad \boldsymbol{c} = -\nabla_0 v_0 - \mu\nabla_0 Q(v^*)\,.$$