Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Leopold Verstraelen

Hypersurfaces for which the third fundamental form is conformal to the quadratic mean form

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.6, p. 599–602.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_6_599_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Geometria differenziale. — *Hypersurfaces for which the third fundamental form is conformal to the quadratic mean form.* Nota di LEOPOLD VERSTRAELEN, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono caratterizzate le ipersuperficie di uno spazio ellittico di cui al titolo, ossia soddisfacenti alla (*). Si dimostra inoltre che le ipersuperficie di quel tipo di uno spazio euclideo che hanno curvatura media costante sono prodotti di sottospazi lineari e sfere.

§ 1. INTRODUCTION

Let M be a condimension 2 submanifold of an elliptic space. Suppose that M is quasi-umbilical w.r.t. 2 orthogonal normal sections ξ and η which are parallel in the normal bundle. Then ξ and η are cylindrical if and only if the third fundamental form of M equals the quadratic mean form [5].

In the present paper first a characterisation is given for the hypersurfaces of Riemannian manifolds whose third fundamental form is conformal to their quadratic mean form, say

$$\text{III} = \rho \text{II}_{\text{H}} . \tag{*}$$

In particular, for $\text{III} = \text{II}_{\text{H}}$, this gives a generalisation of a result of S. Kobayashi and K. Nomizu regarding Ricci flat hypersurfaces of Euclidean spaces [2]. Then it is proved that, in Euclidean space, the hypersurfaces for which (*) holds and which have constant mean curvature are the product submanifolds of linear subspaces of the ambient space with spheres.

§ 2. PRELIMINARIES

Let M^n be an n-dimensional hypersurface of a Riemannian manifold N. Let $b = \{e_A\}$, $(A, B, C \in \{1, 2, \cdots, n+1\})$, be an adapted orthonormal frame associated with $m \in M^n$. Let ω^A and ω_A^B be the dual base forms and the connection forms of M^n w.r.t. b. Then the *metrical fundamental form*, the *connection equations* and the *structure equations* of M^n are respectively given by

$$ds^2 = \sum_i (\omega^i)^2 ; \tag{1}$$

$$\nabla e_A = \omega_A^B \otimes e_B ; \tag{2}$$

$$d \wedge \omega^A = \omega^B \wedge \omega_B^A , \tag{3}$$

$$d \wedge \omega_A^B = \omega_A^C \wedge \omega_C^B + \Omega_A^B \tag{4}$$

(*) Nella seduta dell'11 dicembre 1976.

whereby Ω_A^B are curvature 2-forms of N, and $i, j, k \in \{1, 2, \cdots, n\}$. In particular, if N is a space form of curvature c, then

$$\Omega_A^B = -c\omega^A \wedge \omega^B. \tag{5}$$

One has the relations

$$\omega_A^B = h_{Ai}^B \omega^i ; \tag{6}$$

h_{Ai}^B are the connection coefficients. Since M^n is an integral manifold of

$$\omega^{n+1} = 0 , \tag{7}$$

it follows by exterior differentiation using E. Cartan's lemma that

$$h_{ij}^{n+1} = h_{ji}^{n+1} . \tag{8}$$

The *quadratic mean form*, i.e. the second fundamental form corresponding to the mean curvature vector H, and the *third fundamental form* of M^n are respectively given by

$$II_H = h h_{ij}^{n+1} \omega^i \omega^j; \tag{9}$$

$$III = \sum_i h_{ij}^{n+1} h_{ik}^{n+1} \omega^j \omega^k, \tag{10}$$

whereby $h = \text{trace}\,[h_{ij}^{n+1}]$. If N has constant sectional curvature c, then one has the following formula [4]:

$$\Psi = (n-1)\, c\, ds^2 - III + II_H , \tag{11}$$

where Ψ is the *Ricci form* of M^n. We recall that according to III, II_H or Ψ being proportional to ds^2, respectively being zero, M^n has conformal Gauss map, is pseudo-umbilical or Einstein, respectively is totally geodesic, minimal or Ricci-flat. In particular, (11) shows that the Ricci flat hypersurfaces of Euclidean space are those for which $III = II_H$. Theorem 5.3 (2) of [2] states that for $n \geqq 3$ *the Ricci flat hypersurfaces of Euclidean space are the cylindrical hypersurfaces*, i.e. the hypersurfaces for which zero is a principal curvature with multiplicity $\geqq n-1$ [1].

§ 3. Characterisation of the hypersurfaces for which $III = \rho II_H$

The main purpose of this section is to prove the following.

THEOREM 1. *The third fundamental form of a hypersurface* M^n *of a Riemannian manifold is conformal to the quadratic mean form,* $III = \rho II_H$, *if and only if* p *principal curvatures of* M^n *vanisch and the other principal curvatures are equal. If* M^n *is not totally geodesic* ($p < n$) *then* $\rho = 1/(n-p)$.

Proof. As is well-known we can always chose the vectors e_i in $T_m(M^n)$ such that b is a *principal orthonormal frame*, i.e. such that

$$h_{ij}^{n+1} = 0 \qquad \text{for} \quad i \neq j . \tag{12}$$

Then, denoting the *principal curvatures* h_{ii}^{n+1} of M^n by λ_i, we obtain from (9) and (10) the following expressions for II_H and III:

$$II_H = \left(\sum_j \lambda_j\right) \lambda_i (\omega^i)^2 ; \tag{13}$$

$$III = \lambda_i^2 (\omega^i)^2. \tag{14}$$

Thus, for any function ρ on M^n, (*) holds if and only if

$$\forall_i : \lambda_i (\lambda_i - \rho \sum_j \lambda_j) = 0 . \tag{15}$$

This is equivalent to

$$\forall i \in I = \{1 , 2 ,\cdots, p\} : \lambda_i = 0 \quad ; \quad \forall i \notin I : \lambda_i = \rho \sum_j \lambda_j , \tag{16}$$

whereby $0 \leqq p \leqq n$.

If M^n is not totally geodesic, $0 \leqq p < n$, then (16) implies that

$$\sum_k \lambda_k = (n - p) \rho \sum_j \lambda_j , \tag{17}$$

and consequently

$$\rho = 1/(n - p) . \tag{18}$$

Remark. For the totally geodesic hypersurfaces M^n of N, (*) is trivially satisfied for arbitrary factor of proportionality.

COROLLARY 2. *Let* $M^n, n \geqq 3$, *be a hypersurface of a Riemannian manifold. Then* $III = II_H$ *if and only if* M^n *is cylindrical.*

§ 4. HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPACE FORMS

According to (16), the non-umbilical hypersurfaces M^n for which (*) holds are characterised by

$$\omega_i^{n+1} = 0 \quad , \quad \omega_{i'}^{n+1} = \lambda \omega^{i'} , \tag{19}$$

whereby $\lambda \neq 0$; $i , j , k \in \{1 , 2 ,\cdots, p\}$; $i', j', k' \in \{p + 1 , p + 2 ,\cdots, n\}$ $0 < p < n$. The *mean curvature* of such hypersurfaces is clearly given by

$$\tau = (n - p) \lambda . \tag{20}$$

In the following we consider the *non-umbilical hypersurfaces* M^n *of space forms* N for which (*) *holds* and which have *constant mean curvature*.

By exterior differentiation of (19) we then obtain

$$\omega_{j'}^{i} \wedge \omega^{j'} = \omega_{j}^{i'} \wedge \omega^{j} = 0 . \tag{21}$$

Using E. Cartan's lemma this implies that

$$\forall i , i' : \omega_{i}^{i'} = 0 . \tag{22}$$

Exterior differentiation of (22) shows that essentially c must be zero.

PROPOSITION 3. *The only space forms in which there exist non-umbilical hypersurfaces with constant mean curvature for which* III $= \rho \mathrm{II}_{\mathrm{H}}$ *are the Euclidean spaces.*

Thus we finally consider the hypersurfaces M^n of E^{n+1} for which (19) holds and τ is constant.

We have

$$d \wedge \omega^{i} = \omega^{j} \wedge \omega_{j}^{i} ; \tag{23}$$

$$d \wedge \omega^{i'} = \omega^{j'} \wedge \omega_{j'}^{i'} . \tag{24}$$

Consequently the *distributions* determined respectively by

$$\forall i : \omega^{i} = 0 ; \tag{25}$$

$$\forall i' : \omega^{i'} = 0 , \tag{26}$$

are both *involutive*. Using the connection equations, it is easy to see that the *integral submanifolds* corresponding to (25) and (26) are respectively $(n - p)$-dimensional spheres with radius $1/\lambda$ and p-dimensional linear subspaces of E^{n+1}. Moreover it follows from (22), based on a lemma of J. D. Moore [3], that M^n is actually *product submanifold* of these integral submanifolds.

THEOREM 4. *The non-umbilical hypersurfaces* M^n *of* E^{n+1} *with constant mean curvature* τ *and for which* III $= \rho \mathrm{II}_{\mathrm{H}}$ *are the product submanifolds*

$$E^{p} \times S^{n-p}\left(\frac{n-p}{\tau}\right) \subset E^{p} \times E^{n-p+1}$$

whereby $0 < p < n$.

BIBLIOGRAPHY

[1] B.-Y. CHEN (1973) - *Geometry of submanifolds*, Marcel Dekker Inc., New York.

[2] S. KOBAYASHI and K. NOMIZU (1969) - *Foundations of differential geometry*, vol. II, Interscience Publishers, J. Willey & Sons, New York.

[3] J. D. MOORE (1971) - *Isometric immersions of Riemannian products*, « J. Diff. Geom. », 5.

[4] M. OBATA (1968) - *The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature*, « J. Diff. Geom. », 2.

[5] L. VERSTRAELEN (1975) - *Totaal quasi-ombilicale deelvariëteiten met codimensie twee van een elliptische ruimte*, Doctoraatsbijstelling, Katholieke Universiteit te Leuven.