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Conditions for the identity mapping

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Geometria differenziale. — *Conditions for the identity mapping.*
Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se T è un'applicazione di uno spazio metrico compatto X in sè tale che

$$\rho (Tx , Ty) > \tfrac{1}{2} \{\rho (x , Tx) + \rho (y , Ty)\}$$

per tutti gli $x , y , (x \neq y)$ di X, allora T è l'applicazione identica su X.

In a recent paper, see [1], the following theorem was proved:

THEOREM 1. *If* T *is a mapping of a metric space* X *into itself such that*

$$\rho (Tx , Ty) \geq \tfrac{1}{2} \{\rho (x , Tx) + \rho (y , Ty)\} \tag{1}$$

for all x , y *in* X, *then* T *is the identity mapping on* X.

The theorem does not hold if inequality (1) only holds for distinct x , y in X. This is easily seen by considering a metric space X containing just two points x and y. Define a mapping T on X by

$$Tx = y \quad , \quad Ty = x .$$

Inequality (1) holds for the distinct points x and y, but T is not the identity mapping.

We do however have the following theorem:

THEOREM 2. *If* T *is a mapping of a compact or precompact metric space* X *into itself such that*

$$\rho (Tx , Ty) > \{\rho (x , Tx) + \rho (y , Ty)\} \tag{2}$$

for all distinct x , y *in* X, *then* T *is the identity mapping on* X.

Proof. Let x_0 be an arbitrary point in X and put

$$x_n = T^n x_0$$

for $n = 1 , 2 , \cdots$. Assuming that the sequence $\{x_n : n = 0 , 1 , 2 , \cdots\}$ is a sequence of distinct points, we have

$$\rho (x_n , x_{n+1}) > \tfrac{1}{2} \{\rho (x_{n-1} , x_n) + \rho (x_n , x_{n+1})\}$$

(*) Nella seduta dell'11 dicembre 1976.

and so

$$\rho(x_n, x_{n+1}) > \rho(x_{n-1}, x_n) > \rho(x_0, x_1)$$

for $n = 1, 2, \cdots$. Further

$$\rho(x_n, x_{n+r}) > \tfrac{1}{2}\{\rho(x_{n-1}, x_n) + \rho(x_{n+r-1}, x_{n+r})\} > \rho(x_0, x_1) > 0,$$

since we are supposing that $x_0 \neq x_1$.

It follows that the sequence $\{x_n\}$ can have no Cauchy subsequence, giving a contradiction, since X is either compact or precompact. We must therefore have

$$x_n = x_{n+r}$$

for some $n \geq 0$, $r \geq 1$.

We will now suppose that n is the smallest such n and then that r is the smallest such r. If $n \geq 1$, then by our assumption

$$x_0 \neq x_1, x_{n-1} \neq x_{n+r-1}.$$

It follows that

$$0 = \rho(x_n, x_{n+r}) > \rho(x_0, x_1) > 0,$$

giving a contradiction, and so in fact $n = 0$.

If we suppose that $r > 1$, we have

$$\rho(x_0, x_1) = \rho(x_r, x_1) > \tfrac{1}{2}\{\rho(x_{r-1}, x_r) + \rho(x_0, x_1)\},$$

since $x_{r-1} \neq x_0$. Further

$$\rho(x_{r-1}, x_r) > \rho(x_{r-2}, x_{r-1}) > \rho(x_0, x_1),$$

since $x_{i-1} \neq x_i$ for $i = 1, 2, \cdots, r-1$. It follows that

$$\rho(x_0, x_1) > \rho(x_0, x_1),$$

giving a contradiction. Thus $r = 1$ and so

$$Tx_0 = x_0.$$

Since x_0 was an arbitrary point in X, it follows that T must be the identity mapping on X. This completes the proof of the theorem.

The condition that X be compact or precompact cannot be omitted from Theorem 2. To see this, let

$$X = \{1, 2, \cdots, n, \cdots\}$$

and define a metric on X by putting

$$\rho(m, n) = \begin{cases} \max\{m, n\}, & m \neq n \\ 0, & m = n. \end{cases}$$

Define a mapping T on X by putting

$$Tn = n + 1$$

for $n = 1, 2, \cdots$. Then, if $m \neq n$

$$\rho(Tm, Tn) = 1 + \max\{m, n\}$$

and

$$\rho(m, Tm) + \rho(n, Tn) = 2 + m + n < 2 + 2\max\{m, n\}.$$

It follows that T satisfies inequality (2) for all distinct m, n in X, but T is not the identity mapping on X.

Reference

[1] B. Fisher (1975) – *Mappings on a metric space*, « Boll. Un. Mat. Ital. », *12*, 147–51.