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Characterizations of Einstein Kaehler manifolds and applications

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Geometria differenziale. — *Characterizations of Einstein Kaehler manifolds and applications.* Nota di BANG-YEN CHEN (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Vengono date condizioni sufficienti affinché una varietà compatta di Kaehler o coomologicamente di Einstein-Kaehler sia einsteiniana (Teorema 1, 2); se ne deducono condizioni assicuranti che un'intersezione completa in uno spazio proiettivo complesso risulti uno spazio lineare od un'iperquadrica (Teorema 3).

1. STATEMENT OF RESULTS

Let M be an n-dimensional compact Kaehler manifold. Let $\theta^1, \cdots, \theta^n$ be a local field of unitary coframes with the Kaehler metric g and the Ricci tensor S given by

$$g = \tfrac{1}{2}\,\Sigma\,(\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i)\,,$$

$$S = \tfrac{1}{2}\,\Sigma\,(R_{i\bar{j}}\,\theta^i \otimes \bar{\theta}^j + \bar{R}_{i\bar{j}}\,\bar{\theta}^i \otimes \theta^j)\,,$$

respectively. The fundamental 2-form Φ and the Ricci form γ are then given respectively by

$$\Phi = \frac{\sqrt{-1}}{2}\,\Sigma\theta^i \wedge \bar{\theta}^i \tag{1}$$

$$\gamma = \frac{\sqrt{-1}}{4\pi}\,\Sigma R_{i\bar{j}}\,\theta^i \wedge \bar{\theta}^j. \tag{2}$$

Let $[\sigma]$ denote the cohomology class represented by σ. It is well known that the first Chern class c_1 of M is represented by γ and the last de Rham cohomology group $H^{2n}(M\,;R)$ is generated by $[\Phi^n]$.

We put

$$\omega = [\Phi] \tag{3}$$

and

$$\omega^{n-k}\,c_1^k = a_k\,\omega^n, \qquad k = 0, 1, \cdots, n, \tag{4}$$

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where $\omega^{n-k} c_1^k$ denotes the cup product of ω^{n-k} and c_1^k. We define the k-th scalar curvature ρ_k by

$$\det(\delta_{ij} + t\mathrm{R}_{i\bar{j}}) = \sum_{k=0}^{n} \binom{n}{k} \rho_k t^k,$$

where $\binom{n}{k}$ is the binomial coefficient. It is clear that $\rho_0 = 1$, ρ_1 is the (normalized) scalar curvature and $\rho_n = \det(\mathrm{R}_{ij})$.

We shall prove the following.

THEOREM 1. *Let* M *be an n-dimensional compact Kaehler manifold. If*

(i) $a_k^2 \leqq a_{k-1} a_{k+1}$;

(ii) ρ_k, ρ_{k+1} (or ρ_{k-1}, ρ_k) *are positive constants,*

for some k; $1 \leqq k \leqq n-1$, *then* M *is Einsteinian, where* a_i; $i = k-1, k, k+1$, *are given by* $\omega^{n-i} c_1^i = a_i \omega^n$.

We say that M is cohomologically Einsteinian if $c_1 = b\omega$ for some constant b. As applications of Theorem 1 we shall prove the following [2].

THEOREM 2. *Let* M *be an n-dimensional compact cohomological Einstein Kaehler manifold. If there exists* k, $1 \leqq k \leqq n$, *such that* ρ_{k-1} *and* ρ_k *are positive constants, then* M *is Einsteinian.*

Let $\mathrm{P}_{n+p}(\mathrm{C})$ be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. An n-dimensional algebraic submanifold in $\mathrm{P}_{n+p}(\mathrm{C})$ is called a *complete intersection* if M is given as an intersection of p nonsingular hypersurfaces of $\mathrm{P}_{n+p}(\mathrm{C})$ in general position. A complete intersection is always a Kaehler manifold by considering the induced Kaehler metric from $\mathrm{P}_{n+p}(\mathrm{C})$.

THEOREM 3. *Let* M *be an n-dimensional complete intersection in* $\mathrm{P}_{n+p}(\mathrm{C})$. *If there exists* k, $1 \leqq k \leqq n$, *such that* ρ_{k-1} *and* ρ_k *are positive constants, then* M *is either a linear subspace or a hyperquadric in some* $(n+1)$-*dimensional linear subspace.*

Remark 1. Assumptions (i) and (ii) are essential. For examples: (*a*) Let $\mathrm{M} = \mathrm{P}_k(\mathrm{C}) \times \mathrm{T}^{n-k}$, where T^{n-k} denotes an $(n-k)$-dimensional complex torus with the flat metric. Then $\rho_{k+1} = 0$, ρ_k is constant and $a_k^2 > a_{k-1} a_{k+1} = 0$. (*b*) Let M be *a* algebraic hypersurface of $\mathrm{P}_{n+1}(\mathrm{C})$ with degree $\neq 1, 2$. Then M is cohomologically Einsteinian (see the proof of Theorem 3), in particular, we have $a_k^2 = a_{k-1} a_{k+1}$, but ρ_k, ρ_{k+1} are not constant, simultaneously.

Remark 2. If $k = 1$, the assumption of the constancy of ρ_{k-1} is automatically satisfied. In this case, Theorem 2 and Theorem 3 reduce to results of Kobayashi [5] and Hano [3], respectively.

Remark 3. For hypersurfaces see [1].

2. PROOF OF THEOREMS 1 AND 2

First we prove the following general Lemma.

LEMMA 1. *Let* M *be an n-dimensional compact Kaehler manifold. Then*

$$\int_M \rho_k * 1 = (2\pi)^k a_k \int_M * 1 ,$$

where $\omega^{n-k} c_1^k = a_k \omega^n$ *and* $*$ *denotes the Hodge star operator.*

Proof. Since $\omega^{n-k} c_1^k = a_k \omega^n$, there exists a $(2n-1)$-form η such that

$$\Phi^{n-k} \wedge \gamma^k = a_k \Phi^n + d\eta .$$

From the following identities:

$$* (\Phi^{n-k} \wedge \gamma^k) = \frac{n!\, \rho_k}{(2\pi)^k} , \qquad k = 0, 1, \cdots, n,$$

we find

$$\rho_k = (2\pi)^k a_k + \frac{(2\pi)^k}{n!} * d\eta .$$

Thus by taking integration of both sides of this equation over M and by using the identity $(* d\eta) * 1 = d\eta$, we get the lemma.

Now we return to the proof of Theorem 1.

From assumption (ii) ρ_k and ρ_{k+1} (or ρ_{k-1} and ρ_k) are constant, then from Lemma 1 we find

$$\int_M \rho_k^2 * 1 = (2\pi)^k a_k \int_M \rho_k * 1 = (2\pi)^{2k} a_k^2 \int_M * 1 ,$$

$$\int_M \rho_{k-1} \rho_{k+1} * 1 = (2\pi)^{2k} a_{k-1} a_{k+1} \int_M * 1 .$$

Then by assumption (i) we find

$$\int_M (\rho_k^2 - \rho_{k-1} \rho_{k+1}) * 1 \leqq 0 . \tag{5}$$

On the other hand, from the definition of ρ_k and a well-known inequality on elementary symmetric functions we have

$$\rho_k^2 - \rho_{k-1} \rho_{k+1} \geqq 0 , \tag{6}$$

where the equality holds if and only if (R_{ij}) is proportional to the identity matrix, i.e., M is Einsteinian. Thus from (5) and (6) we see that $\rho_k^2 = \rho_{k-1} \rho_{k+1}$ and M is Einsteinian. This proves Theorem 1.

If M is cohomologically Einsteinian, then we have

$$c_1 = b\omega$$

for some constant b. Then, by (4), we have

$$a_k = b^k, \qquad k = 1, \cdots, n.$$

From these we find $a_k^2 = a_{k-1}\, a_{k+1}$. Thus Theorem 2 follows immediately from Theorem 1.

3. PROOF OF THEOREM 3

Let M be a complete intersection in $P_{n+p}(C)$ given as the intersection of p nonsingular hypersurfaces $M_1, \cdots, M_p$, in general position. Let d_α denote the degree of M_α ; $a = 1, \cdots, p$. Then by a theorem of Riemann-Roch-Hirzebruch ([4, p. 159]) the first Chern class c_1 of M is given by

$$c_1 = \frac{n + p - 1 - \Sigma d_\alpha}{4\pi}\,\omega .$$

This shows that M is cohomologically Einsteinian. Thus, if ρ_{k-1} and ρ_k are constants for some k, $1 \leqq k \leqq n$, then Theorem 2 implies that M in Einsteinian. Thus, by a result of Hano, we see that M is either a linear subspace or a hyperquadric in some $(n+1)$-dimensional linear subspace. This completes the proof of the theorem.

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