Lu-San Chen, Cheh-Chih Yeh

A note on n-th order differential inequalities

Accademia Nazionale dei Lincei

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Equazioni differenziali ordinarie. — A note on n-th order differential inequalities \(^{(1)}\). Nota di Lu-San Chen e Cheh-Chih Yeh, presentata \(^{(2)}\) dal Socio G. Sansone.

RIASSUNTO. — Gli Autori trovano un teorema di confronto tra gli integrali oscillatori di due equazioni funzionali ordinarie.

1. We consider the following functional differential equation and inequalities:

\[
\begin{align*}
(1) & \quad L_n x + H(t, x) \leq 0, \\
(2) & \quad L_n x + H(t, x) = 0, \\
(3) & \quad L_n x + H(t, x) \geq 0,
\end{align*}
\]

where \( n \) is even and \( L_n \) is defined by

\[
L_0 x(t) = x(t), \quad L_i x(t) = r^i(t)L_{i-1} x(t), \quad i = 1, 2, \ldots, n, r_n(t) = 1.
\]

We first show that the existence of a positive solution of the inequality (1) implies the same fact for the equation (2) provided that \( H(t, x) \) is positive and increasing for positive \( x \). Similarly, we can prove that the existence of a negative solution of the inequality (3) implies the same fact for the equation (2) provided that \( H(t, x) \) is negative and increasing for negative \( x \). Using these results, we obtain a criterion for the oscillation of (2) via comparison with another equation of the same type, which is oscillatory. The technique used is an adaptation of that of Kartsatos \([1]\) which concerns the particular case.

\[
r_1(t) = r_2(t) = \cdots = r_{n-1}(t) = 1.
\]

A function is said to be oscillatory if it has an unbounded set of zeros. A bounded nonoscillatory function \( h(t) \) is said to be of class F if there exists a \( T > 0 \) such that for \( t \geq T \)

\[
(-1)^{i+1} h(t)L_i h(t) \geq 0, \quad i = 1, 2, \cdots, n - 1.
\]

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\(^{(2)}\) Nella seduta del 13 novembre 1976.
Throughout this note, the following conditions always hold:

(i) each \( r_1(t) \) is a continuous and positive function on \([\tau, \infty)\) and

\[
\int_{\tau}^{\infty} \frac{dt}{r_1(t)} = \infty, \quad i = 1, 2, \ldots, n - 1,
\]

(ii) \( H(t, x) \in C \left( (\tau, \infty) \times \mathbb{R}, \mathbb{R} \right) \), \( xH(t, x) > 0 \) for \( x \neq 0 \) and \( H(t, x) \) is nondecreasing with respect to \( x \).

2. In order to prove that the existence of a positive solution of (1) implies the same fact for (2), we need the following Lemma, which is due to Kusano and Onose [2].

**Lemma 1.** Let \( z(t) \) be a bounded nonoscillatory solution of (1). Then \( z(t) \) belongs to the class \( F \).

The following Lemma is an improved version of Kartsatos’ Lemma [1].

**Lemma 2.** Let \( z(t) \) be a bounded positive solution of (1) for \( t \geq T \). If \( x_0 \) is such that \( 0 < x_0 < z(T) \), then there exists a solution \( x(t) \) of (2) such that \( x(T) = x_0 \), \( x(t) \in F \) and for \( t \geq T \)

\[
0 < x^{(i)}(t) \leq z^{(i)}(t), \quad i = 0, 1,
\]

\[
o > (-1)^i L_i x(t) \geq L_i z(t), \quad i = 2, 3, \ldots, n.
\]

**Proof.** From Lemma 1, \( z(t) \in F \). Integrating (1) from \( t \) to \( u (\geq t \geq T) \), we have

\[
L_{n-1}z(t) = \frac{r_{n-1}(t)}{L_{n-1}z(t)} \geq L_{n-1}z(u) + \int_t^u H(s, z(s)) \, ds \geq
\]

\[
\int_t^u H(s, z(s)) \, ds,
\]

which implies for \( t \geq T \)

\[
(L_{n-2}z(t))' \geq \frac{1}{r_{n-1}(t)} \int_t^\infty H(s, z(s)) \, ds.
\]

Integrating it from \( t \) to \( u (\geq t \geq T) \) yields

\[
r_{n-2}(u) (L_{n-3}z(u))' - r_{n-2}(t) (L_{n-3}z(t))' \geq
\]

\[
\geq \int_t^u \frac{1}{r_{n-1}(u)} \int_{u_1}^\infty H(s, z(s)) \, ds \, du_1,
\]
and since
\[(L_{n-3} z(\mu))' \leq 0 ,\]

\[r_{n-2}(t) (L_{n-3} z(\mu))' \leq - \int_{t}^{\infty} \int_{u_1}^{\infty} \frac{1}{r_{n-1}(\mu_1)} H(s, z(s)) \, ds \, du_1 .\]

Similarly, we obtain
\[r_1(t) z'(t) \geq \int_{t}^{\infty} \int_{u_{n-2}}^{\infty} \int_{u_{n-3}}^{\infty} \cdots \int_{u_2}^{\infty} \int_{u_1}^{\infty} \frac{1}{r_{n-1}(\mu_1)} H(s, z(s)) \, ds \, du_1 \cdots du_{n-2} .\]

Hence
\[(4) \quad z(t) \geq z(T) + \int_{T}^{t} \int_{u_{n-1}}^{\infty} \int_{u_{n-2}}^{\infty} \cdots \int_{u_2}^{\infty} \int_{u_1}^{\infty} \frac{1}{r_{n-1}(\mu_1)} H(s, z(s)) \, ds \, du_1 \cdots du_{n-1} = z(T) + \varphi(t, z) , \quad t \geq T .\]

Let
\[x_0(t) = z(t) , \quad x_{n+1}(t) = x_0 + \varphi(t, x_n) , \quad n = 1, 2, \ldots .\]

From (4) we obtain by mathematical induction that
\[0 < x_n(t) \leq z(t) , \quad x_{n+1}(t) \leq x_n(t) ,\]
for \(t \geq T\) and \(n = 0, 1, \ldots\). Therefore, there exists a function \(x(t)\) such that \(\lim_{n \to \infty} x_n(t) = x(t)\), and applying Lebesgue's theorem on monotone convergence we get
\[x(t) = x_0 + \varphi(t, x) .\]

It follows easily that \(x(t)\) has the desired properties.

Similarly, we have the following

**Lemma 3.** Let \(z(t)\) be a bounded solution of (3), which is negative for \(t \geq T\). If \(x_0\) is such that \(0 < x_0 < z(T)\), then there exists a solution \(x(t)\) of (2) such that the conclusion of Lemma 2 holds.
3. Using the above three lemmas, we can prove the following theorem.

**Theorem.** Let the functions \( H_i(t, u), i = 1, 2 \) be defined on \([\tau, \infty) \times \mathbb{R}\), increasing with respect to \( u \), and \( uH_i(t, u) > 0 \) for \( u \neq 0 \). Let there exist an oscillatory function \( P(t) \) such that for \( t \geq \tau \)

\[
L_n P(t) \equiv Q(t)
\]

and \( \lim_{t \to \infty} P(t) = 0 \). If

\[
H_1(t, u) \leq H_2(t, u), \quad t \geq \tau, \quad u > 0
\]

\[
H_1(t, u) \geq H_2(t, u), \quad t \geq \tau, \quad u < 0
\]

and every bounded solution of

\[
L_n x + H_1(t, x) = Q(t)
\]

is oscillatory, then every bounded solution of

\[
L_n x + H_2(t, x) = Q(t)
\]

is also oscillatory.

**Proof.** Let (6) be nonoscillatory. Then there exists at least one bounded nonoscillatory solution \( z(t) \) of (6). Let \( z(t) > 0 \) for \( t \geq T \). Then \( u(t) \equiv z(t) - P(t) \) is an eventually positive solution of the equation

\[
L_n u(t) + H_1(t, u(t) + P(t)) = 0.
\]

Since \( z(t) = u(t) + P(t) > 0 \) for \( t \geq T \), which implies

\[
L_n u(t) < 0 \quad \text{for} \quad t \geq T.
\]

Hence \( u(t) \) has to be eventually of constant sign. If \( u(t) < 0 \) for \( t \) large enough, then \( P(t) > -u(t) > 0 \) for \( t \) large enough, a contradiction to the oscillatory character of \( P(t) \). Hence \( u(t) > 0 \) eventually. From Lemma 1, there is a \( T_1 \geq T \) such that for \( t \geq T_1 \)

\[
u(t) > 0, \quad u'(t) > 0.
\]

Let \( T_1 \) be large enough so that we also have \( |P(t)| < \epsilon < u(T_1) \) for \( t \geq T_1 \), where \( \epsilon \) is a positive constant. Hence for \( t \geq T_1 \),

\[
L_n u(t) + H_1(t, u(t) + P(t)) \leq L_n u(t) + H_2(t, u(t) + P(t)) = 0.
\]

It follows, from Lemma 1, that

\[
L_n u(t) + H_1(t, u(t) + P(t)) \leq 0.
\]
has a solution $u(t) \in F$. Now it is easy to show the existence of a positive solution to the integral equation

$$(7) \quad \vartheta(t) = c + \varphi(t, \vartheta + P), \quad t \geq T_1.$$ 

We only have to note that if

$$\vartheta_0(t) = u(t)$$

$$\vartheta_{n+1}(t) = c + \varphi(t, \vartheta_{n} + P), \quad n = 1, 2, \ldots,$$

then $H(t, \vartheta_n(t) + P(t)) > 0$ for each $n$, because $\vartheta_n(t) + P(t) > c + P(t) > 0$ for $t \geq T_1$. Differentiating $(7)$ $n$ times, we obtain

$$L_n \vartheta(t) + H(t, \vartheta(t) + P(t)) = 0.$$ 

Letting $y(t) = \vartheta(t) + P(t)$, we get for $t \geq T_1$

$$(8) \quad L_n y(t) + H(t, y(t)) = Q(t).$$

Since $\vartheta(t) + P(t) \geq c + P(t) > 0$, it follows from Lemma 2 that $(8)$ has an eventually positive solution, a contradiction. Similarly using Lemmas 1 and 3, we can prove the case for an eventually negative solution of equation $(6)$.

References
