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## A note on n-th order differential inequalities

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**Equazioni differenziali ordinarie.** — *A note on n-th order differential inequalities* (*). Nota di LU-SAN CHEN e CHEH-CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori trovano un teorema di confronto tra gli integrali oscillatori di due equazioni funzionali ordinarie.

1. We consider the following functional differential equation and inequalities:

$$L_n x + H(t, x) \leq 0, \tag{1}$$

$$L_n x + H(t, x) = 0, \tag{2}$$

$$L_n x + H(t, x) \geq 0, \tag{3}$$

where $n$ is even and $L_n$ is defined by

$$L_0 x(t) = x(t), L_i x(t) = r_i(t) (L_{i-1} x(t))', \qquad i = 1, 2, \cdots, n, r_n(t) = 1.$$

We first show that the existence of a positive solution of the inequality (1) implies the same fact for the equation (2) provided that $H(t, x)$ is positive and increasing for positive $x$. Similarly, we can prove that the existence of a negative solution of the inequality (3) implies the same fact for the equation (2) provided that $H(t, x)$ is negative and increasing for negative $x$. Using these results, we obtain a criterion for the oscillation of (2) via comparison with another equation of the same type, which is oscillatory. The technique used is an adaptation of that of Kartsatos [1] which concerns the particular case.

$$r_1(t) = r_2(t) = \cdots = r_{n-1}(t) = 1.$$

A function is said to be *oscillatory* if it has an unbounded set of zeros. A bounded nonoscillatory function $h(t)$ is said to be of class F is there exists a $T > 0$ such that for $t \geq T$

$$(-1)^{i+1} h(t) L_i h(t) \geq 0, \qquad i = 1, 2, \cdots, n-1.$$

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Throughout this note, the following conditions always hold:

(i) *each* $r_i(t)$ *is a continuous and positive function on* $[\tau, \infty)$ *and*

$$\int_{\tau}^{\infty} \frac{dt}{r_i(t)} = \infty, \qquad i = 1, 2, \cdots, n-1,$$

(ii) $H(t, x) \in C[[\tau, \infty) \times R, R]$, $xH(t, x) > 0$ *for* $x \neq 0$ *and* $H(t, x)$ *is nondecreasing with respect to* $x$.

2. In order to prove that the existence of a positive solution of (1) implies the same fact for (2), we need the following Lemma, which is due to Kusano and Onose [2].

LEMMA 1. *Let* $z(t)$ *be a bounded nonoscillatory solution of* (1). *Then* $z(t)$ *belongs to the class* F.

The following Lemma is an improved version of Kartsatos' Lemma [1].

LEMMA 2. *Let* $z(t)$ *be a bounded positive solution of* (1) *for* $t \geq T$. *If* $x_0$ *is such that* $0 < x_0 < z(T)$, *then there exists a solution* $x(t)$ *of* (2) *such that* $x(T) = x_0$, $x(t) \in F$ *and for* $t \geq T$

$$0 < x^{(i)}(t) \leq z^{(i)}(t), \qquad i = 0, 1,$$

$$0 > (-1)^i L_i x(t) \geq L_i z(t), \qquad i = 2, 3, \cdots, n.$$

*Proof.* From Lemma 1, $z(t) \in F$. Integrating (1) from $t$ to $u\ (\geq t \geq T)$, we have

$$L_{n-1} z(t) = r_{n-1}(t)(L_{n-2} z(t))' \geq L_{n-1} z(u) + \int_t^u H(s, z(s))\, ds \geq$$

$$\geq \int_t^u H(s, z(s))\, ds,$$

which implies for $t \geq T$

$$(L_{n-2} z(t))' \geq \frac{1}{r_{n-1}(t)} \int_t^{\infty} H(s, z(s))\, ds.$$

Integrating it from $t$ to $u\ (\geq t \geq T)$ yields

$$r_{n-2}(u)(L_{n-3} z(u))' - r_{n-2}(t)(L_{n-3} z(t))' \geq$$

$$\geq \int_t^u \frac{1}{r_{n-1}(u_1)} \int_{u_1}^{\infty} H(s, z(s))\, ds\, du_1,$$

and since

$$(L_{n-3}\, z(u))' \leq 0 ,$$

$$r_{n-2}(t)\,(L_{n-3}\, z(t))' \leq -\int_t^{\infty} \frac{1}{r_{n-1}(u_1)} \int_{u_1}^{\infty} H(s , z(s))\, ds\, du_1 .$$

Similary, we obtain

$$r_1(t)\, z'(t) \geq \int_t^{\infty} \frac{1}{r_2(u_{n-2})} \int_{u_{n-2}}^{\infty} \frac{1}{r_3(u_{n-3})} \int_{u_{n-3}}^{\infty} \cdots \int_{u_2}^{\infty} \frac{1}{r_{n-1}(u_1)} \cdot$$

$$\cdot \int_{u_1}^{\infty} H(s , z(s))\, ds\, du_1 \cdots du_{n-2} .$$

Hence

$$(4) \qquad z(t) \geq z(T) + \int_T^{t} \frac{1}{r_1(u_{n-1})} \int_{u_{n-1}}^{\infty} \frac{1}{r_2(u_{n-2})} \int_{u_{n-2}}^{\infty} \cdots \int_{u_2}^{\infty} \frac{1}{r_{n-1}(u_1)} \cdot$$

$$\cdot \int_{u_1}^{\infty} H(s , z(s))\, ds\, du_1 \cdots du_{n-1} \equiv$$

$$\equiv z(T) + \varphi(t , z) , \qquad t \geq T .$$

Let

$$x_0(t) = z(t)$$

$$x_{n+1}(t) = x_0 + \varphi(t , x_n) , \qquad n = 1 , 2 , \cdots .$$

From (4) we obtain by mathematical induction that

$$0 < x_n(t) \leq z(t) ,$$

$$x_{n+1}(t) \leq x_n(t) ,$$

for $t \geq T$ and $n = 0 , 1 , \cdots$. Therefore, there exists a function $x(t)$ such that $\lim_{n\to\infty} x_n(t) = x(t)$, and applying Lebesgue's theorem on monotone convergence we get

$$x(t) = x_0 + \varphi(t , x) .$$

It follows easily that $x(t)$ has the desired properties.

Similarly, we have the following

LEMMA 3. *Let $z(t)$ be a bounded solution of (3), which is negative for $t \geq T$. If $x_0$ is such that $0 < x_0 < z(T)$, then there exists a solution $x(t)$ of (2) such that the conclusion of Lemma 2 holds.*

3. Using the above three lemmas, we can prove the following theorem.

THEOREM. *Let the functions* $H_i(t, u)$, $i = 1, 2$ *be defined on* $[\tau, \infty) \times R$, *increasing with respect to* $u$, *and* $uH_i(t, u) > 0$ *for* $u \neq 0$. *Let there exist an oscillatory function* $P(t)$ *such that for* $t \geq \tau$

$$L_n P(t) \equiv Q(t)$$

*and* $\lim_{t\to\infty} P(t) = 0$. *If*

$$H_1(t, u) \leq H_2(t, u), \quad t \geq \tau, \qquad u > 0$$

$$H_1(t, u) \geq H_2(t, u), \quad t \geq \tau, \qquad u < 0$$

*and every bounded solution of*

(5) $$L_n x + H_1(t, x) = Q(t)$$

*is oscillatory, then every bounded solution of*

(6) $$L_n x + H_2(t, x) = Q(t)$$

*is also oscillatory.*

*Proof.* Let (6) be nonoscillatory. Then there exists at least one bounded nonoscillatory solution $z(t)$ of (6). Let $z(t) > 0$ for $t \geq T$. Then $u(t) \equiv \equiv z(t) - P(t)$ is an eventually positive solution of the equation

$$L_n u(t) + H_2(t, u(t) + P(t)) = 0 .$$

Since $z(t) = u(t) + P(t) > 0$ for $t \geq T$, which implies

$$L_n u(t) < 0 \qquad \text{for} \quad t \geq T .$$

Hence $u(t)$ has to be eventually of constant sign. If $u(t) < 0$ for $t$ large enough, then $P(t) > -u(t) > 0$ for $t$ large enough, a contradiction to the oscillatory character of $P(t)$. Hence $u(t) > 0$ eventually. From Lemma 1, there is a $T_1 \geq T$ such that for $t \geq T_1$

$$u(t) > 0 \quad , \quad u'(t) > 0 .$$

Let $T_1$ be large enough so that we also have $|P(t)| < c < u(T_1)$ for $t \geq T_1$, where $c$ is a positive constant. Hence for $t \geq T_1$,

$$L_n u(t) + H_1(t, u(t) + P(t)) \leq L_n u(t) + H_2(t, u(t) + P(t)) = 0 .$$

It follows, from Lemma 1, that

$$L_n u(t) + H_1(t, u(t) + P(t)) \leq 0$$

has a solution $u(t) \in F$. Now it is easy to show the existence of a positive solution to the integral equation

$$\vartheta(t) = c + \varphi(t, \vartheta + P), \qquad t \geq T_1. \tag{7}$$

We only have to note that if

$$\vartheta_0(t) = u(t)$$
$$\vartheta_{n+1}(t) = c + \varphi(t, \vartheta_n + P), \qquad n = 1, 2, \cdots,$$

then $H(t, \vartheta_n(t) + P(t)) > 0$ for each $n$, because $\vartheta_n(t) + P(t) > c + P(t) > 0$ for $t \geq T_1$. Differentiating (7) $n$ times, we obtain

$$L_n \vartheta(t) + H(t, \vartheta(t) + P(t)) = 0.$$

Letting $y(t) = \vartheta(t) + P(t)$, we get for $t \geq T_1$

$$L_n y(t) + H(t, y(t)) = Q(t). \tag{8}$$

Since $\vartheta(t) + P(t) \geq c + P(t) > 0$, it follows from Lemma 2 that (8) has an eventually positive solution, a contradiction. Similarly using Lemmas 1 and 3, we can prove the case for an eventually negative solution of equation (6).

## REFERENCES

[1] A. G. KARTSATOS (1975) – *On n-th order differential inequalities*, « J. Math. Anal. Appl. », 52, 1–9.

[2] T. KUSANO and H. ONOSE (1976) – *Remarks on the oscillatory behavior of solution of functional differential equations with deviating arguments*, « Hiroshima Math. J. », 6, 183–189.