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Gheorghe Morosanu

Asymptotic behaviour of resolvent for a monotone mapping in a Hilbert space

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Analisi funzionale. — *Asymptotic behaviour of resolvent for a monotone mapping in a Hilbert space.* Nota di GHEORGHE MOROSANU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota sono dimostrati alcuni risultati riguardanti il comportamento asintotico delle risolventi di un operatore massimale monotono in uno spazio di Hilbert.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space with scalar product denoted by $(\cdot, \cdot)$ and norm denoted by $\|\cdot\|$. Let A be a monotone subset of $H \times H$, i.e. for every $[x_i, y_i] \in A$, $i = 1, 2$, we have $(y_1 - y_2, x_1 - x_2) \geq 0$. A is said to be maximal monotone if it is monotone and there is no monotone set $\tilde{A} \subset H \times H$ such that $A \subset \tilde{A}$, $A \neq \tilde{A}$. A may be considered as a multivalued mapping defined on $D(A) = \{x ; [x, y] \in A \text{ for some } y \in H\}$ with values in $R(A) = \{y ; [x, y] \in A \text{ for some } x \in H\}$, defining: $Ax = \{y ; [x, y] \in A\}$. We put

$$J_t x = (I + tA)^{-1} x, \; A_t x = t^{-1}(x - J_t x), \qquad \text{for} \quad x \in R(I + tA), \; t > 0.$$

The purpose of this paper is to study the asymptotic behaviour, as $t \to \infty$, of the trajectory $t \to J_t x$. Theorems 2.1 and 2.2 state the main results of asymptotic convergence. Moreover, we shall see that these results are analogous to those concerning the asymptotic behaviour of a semigroup of nonlinear contractions.

We shall collect, for easy reference, in the following lemma some elementary properties of J_t and A_t (for the proof see [3]).

LEMMA 1.1. *Let* A *be a monotone subset of* $H \times H$ *and let* $t > 0$. *Then*

(*a*) $\|J_t x - J_t y\| \leq \|x - y\|$, *for all* $x, y \in R(I + tA)$,

(*b*) A_t *is monotone in* $H \times H$ *and Lipschitz on* $R(I + tA)$, *with Lipschitz constant* $2t^{-1}$,

(*c*) $A_t x \in A J_t x$, *for every* $x \in R(I + tA)$,

(*d*) $\|A_t x\| \leq |Ax|$, *for every* $x \in D(A) \cap R(I + tA)$, *where* $|Ax| = \inf\{y ; y \in Ax\}$.

Next, we consider that A is maximal monotone and let S be the semigroup of contractions generated by A

$$S(t)x : [0, \infty) \times \overline{D(A)} \to \overline{D(A)}.$$

(*) Nella seduta dell'11 dicembre 1976.

We put

$$F = \{x \in \overline{D(A)}\,;\, S(t)\,x = x\,, \quad \text{for every} \quad t \geq 0\}.$$

It is easy to see that $F = A^{-1}0$ and because A^{-1} is maximal monotone it follows that F is a closed convex subset of $\overline{D(A)}$. It is well known that if A is maximal monotone, then $R(I + tA) = H$, for every $t > 0$. In this case J is defined on $[0, \infty) \times H$.

The following relation is immediate

$$\|J_t x - J_s x\| = \|J_s(st^{-1}x + (t-s)\,t^{-1}J_t x) - J_s x\| \leq t^{-1}|t-s| \cdot \|J_t x - x\|,$$

for every $x \in H$ and $t, s > 0$.

Therefore, for any $x \in H$, the function $t \to J_t x$ is continuous in $t > 0$.

We also note that

$$F = \{x \in H\,;\, J_t x = x\,, \quad \text{for every} \quad t \geq 0\}.$$

Hence, S and J have the same fixed points.

2. Results

THEOREM 2.1. *Let* A *be a maximal monotone set of* $H \times H$. *Then*

$$\lim_{t \to \infty} J_t x/t = -a^0, \quad \textit{for every} \quad x \in H$$

where a^0 *is the element of minimum norm of* $\overline{R(A)}$.

Proof. We shall prove that

$$\lim_{t \to \infty} A_t x = a^0, \quad \text{for every} \quad x \in H. \tag{2.1}$$

Since A is monotone, it follows (see [3], Prop. 3.5, Chap. II) that the function $t \to \|A_t x\|$ is monotone nonincreasing, $0 < t < \infty$, $x \in H$. Therefore, there exists

$$\lim_{t \to \infty} \|A_t x\| = \rho(x), \quad \text{for any} \quad x \in H.$$

But

$$\|A_t y\| \leq \|A_t y - A_t x\| + \|A_t x\| \leq 2\,t^{-1}\|x - y\| + \|A_t x\| \quad \text{for every} \quad x, y \in H.$$

This implies that

$$\rho(x) = \rho(y) = \rho, \quad \text{for every} \quad x, y \in H.$$

Hence

$$\lim_{t \to \infty} \|A_t x\| = \rho, \quad \text{for every} \quad x \in H. \tag{2.2}$$

Since

$$\|A_t x\| \leq |Ax|, \qquad \forall x \in D(A) \cap R(I + tA) = D(A),$$

it follows that

$$\rho \leq \|a^0\|. \tag{2.3}$$

But

$$A_t x \in AJ_t x \subset \overline{R(A)}, \qquad \forall x \in H. \tag{2.4}$$

Obviously

$$\|a^0\| \leq \|A_t x\|, \qquad \forall x \in H$$

and letting $t \to \infty$, one obtains

$$\|a^0\| \leq \rho. \tag{2.5}$$

It is well known (see [7]) that if $C \subset H$ is closed convex with the element of minimum norm denoted by v and $u_n \in C$, such that $\|u_n\| \to \|v\|$, then $u_n \to v$. Since A is a maximal monotone set of $H \times H$, it follows that $\overline{R(A)}$ is a convex set. Therefore, from (2.2), (2.3), (2.5) and (2.4), one obtains (2.1) and the theorem is proved.

COROLLARY 2.1. *Let* A *be a closed monotone set of* $H \times H$, *and*

$$\overline{\text{conv}\, D(A)} \subset R(I + tA), \qquad \textit{for every} \quad t > 0.$$

Then

$$\lim_{t \to \infty} J_t x/t = -a^0, \qquad \textit{for every} \quad x \in \overline{D(A)},$$

where a^0 *is the element of minimum norm in* $\overline{\text{conv}\, R(A)}$ *and* $a^0 \in \overline{R(A)}$.

Proof. Under the assumptions of this corollary, it follows in virtue of a result due to Brézis and Pazy (see [3], p. 186), that A has a unique extension to a maximal monotone set $\tilde{A} \subset H \times H$, $A \subset \tilde{A}$, satisfying $D(A) = D(\tilde{A})$ and $(I + tA)^{-1} x = (I + t\tilde{A})^{-1} x$, for every $x \in \overline{D(A)} = \overline{\text{conv}\, D(A)}$.

From Theorem 2.1 one deduces that there exists

$$\lim_{t \to \infty} J_t x/t = -a^0, \qquad \text{for every} \quad x \in \overline{D(A)},$$

where a^0 is the element of minimum norm in $\overline{R(\tilde{A})}$ and $a^0 \in \overline{R(A)}$ (see the proof of Theorem 1 in [4]). Since

$$\overline{R(A)} \subset \overline{\text{conv}\, R(A)} \subset \overline{R(\tilde{A})},$$

we observe that a^0 is the element of minimum norm in $\overline{\text{conv}\, R(A)}$, thereby the proof is complete.

Remark 2.1. Theorem 2.1 is a result analogous to that proved by M. G. Crandall (see [8], Prop. 3.9) for a semigroup of nonlinear contractions, i.e. if A is maximal monotone in $H \times H$ then, for every $x \in \overline{D(A)}$, $\lim_{t\to\infty} S(t)\,x/t = -a^0$, where S is the semigroup generated by A.

The following theorem may be compared with a result proved for linear operators by H. Komatsu [6].

THEOREM 2.2. *Let* A *be a maximal monotone set of* $H \times H$. *Then there exists*

$$\lim_{t\to\infty} J_t x = u_x, \qquad \textit{for every} \quad x \in H \tag{2.6}$$

if and only if the set of fixed points F *is nonempty. When the limit exists, it coincides with* Px (*the projection of* x *on* F)

Proof. First, we suppose that $F \neq \Phi$. Clearly

$$(J_t x - J_t y, x - y) \geq \|J_t x - J_t y\|^2, \qquad \text{for every} \quad x, y \in H.$$

If $y \in F$, then the above inequality yelds

$$(x - J_t x, y - J_t x) \leq 0, \qquad \text{for every} \quad x \in H \quad \text{and} \quad y \in F.$$

This implies

$$\|x - J_t x\| \leq \|x - y\|, \qquad \text{for every} \quad x \in H, y \in F. \tag{2.7}$$

Therefore

$$\|x - J_t x\| \leq \|x - Px\|, \qquad \text{for every} \quad x \in H. \tag{2.8}$$

It follows that there exists a sequence $t_n \to \infty$ such that $J_{t_n} x$ converges weakly to an element $u_x \in H$. Since A is monotone, we have

$$(w - A_{t_n} x, v - J_{t_n} x) \geq 0, \qquad \forall [v, w] \in A$$

and, letting $t_n \to \infty$, we obtain

$$(w, v - u_x) \geq 0, \qquad \forall [v, w] \in A. \tag{2.9}$$

From (2.9) and the maximality of A one deduces that $u_x \in D(A)$ and $0 \in Au_x$. Hence, $u_x \in F$. Taking into account (2.7) we deduce that $J_{t_n} x$ converges strongly to u_x, as $t_n \to \infty$. From (2.8), it follows that $u_x = Px$. Since every subsequence of $J_t x$ converges to the unique element Px, it follows that (2.6) holds.

Conversely, if we suppose that (2.6) holds, then, as above, we obtain

$$(w, v - u_x) \geq 0, \qquad \forall [v, w] \in A,$$

which implies $u_x \in F$, i.e. $F \neq \Phi$ and the proof of the theorem is complete.

Remark 2.2. Taking in Theorem 2.2 $x = 0$, we obtain

$$\lim_{t \to \infty} J_t 0 = u^0 \Longleftrightarrow F \neq \Phi ,$$

where u^0 is the element of minimum norm in F.

Remark 2.3. Theorem 2.2 is analogous to a recent asymptotic result for nonlinear semigroups proved by A. Pazy ([8], Theorem 2.1).

If $F \neq \Phi$, Baillon and H. Brézis [1] proved the weak convergence of $\sigma(t)\, x = t^{-1} \int_0^t S(s)\, x\, ds$, as $t \to \infty$, to a fixed point of S. If, in addition, S is a semigroup of odd contractions, Baillon [2] has proved the strong convergence of $\sigma(t)\, x$. An analogous result is the following:

COROLLARY 2.2. *Let* A *be a maximal monotone set in* H×H *and suppose that* $F \neq \Phi$. *Then*

$$\lim_{t \to \infty} t^{-1} \int_0^t J_s x \, ds = Px .$$

Proof. We shall prove that

$$\lim_{t \to \infty} t^{-1} \int_0^t \| J_s x - Px \| \, ds = 0 . \tag{2.10}$$

We take an arbitrary sequence $b_n \to \infty$ and we denote

$$a_n = \int_0^{b_n} \| J_s x - Px \| \, ds .$$

Without any loss of generality we may assume that b_n is increasing. We have

$$\lim_{n \to \infty} (a_{n+1} - a_n)(b_{n+1} - b_n)^{-1} = \lim_{n \to \infty} (b_{n+1} - b_n)^{-1} \int_{b_n}^{b_{n+1}} \| J_s x - Px \| \, ds =$$

$$= \lim_{n \to \infty} \| J_{t_n} x - Px \| = 0 ,$$

where $t_n \in (b_n , b_{n+1})$ is chosen by the mean theorem. We conclude that (2.10) holds, as claimed.

Taking into account the definition of a bounded semigroup of contractions as in [8], we give:

DEFINITION 2.1. If A is maximal monotone set in H×H, then J is said to be bounded if for every $x \in H$, the trajectory $t \to J_t x$ is bounded in H.

By (*a*), Lemma 1.1, it follows that if one trajectory is bounded then all trajectories are bounded.

COROLLARY 2.3. *Let* A *be a maximal monotone set in* H × H *and let* S *be the semigroup of contractions generated by* A. *Then*

$$0 \in R(A) \Longleftrightarrow S \quad \textit{is bounded} \Longleftrightarrow J \textit{ is bounded},$$

$$0 \notin R(A) \Longleftrightarrow \lim_{t\to\infty} \|J_t x\| = \infty, \qquad \forall x \in H;$$

$$\textit{if} \quad 0 \in \overline{R(A)} \setminus R(A), \quad \textit{then} \quad \lim_{t\to\infty} t^{-1} \|J_t x\| = 0, \qquad \forall x \in H;$$

$$\textit{if} \quad 0 \notin \overline{R(A)}, \quad \textit{then} \quad \lim_{t\to\infty} t^{-1} \|J_t x\| = \alpha > 0, \qquad \forall x \in H.$$

Proof. The equivalence

$$0 \in R(A) \Longleftrightarrow S \quad \text{is bounded}$$

was first established by Crandall and Pazy [5] and next, using a more simple argument, by Pazy [8]. The other conditions are immediate.

Remark 2.4. The above corollary is a result similar to Theorem 2 [4] concerning the behaviour of a semigroup generated by the closed monotone set. See also Prop. 3.10 and Corollary 3.11 in [8].

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