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On a class of integro-differential equations

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Analisi matematica. — *On a class of integro–differential equations.* Nota VI di MEHMET NAMIK OĞUZTÖRELI (*),(**), presentata (***) dal Socio M. PICONE.

RIASSUNTO. — In questo lavoro si studia un'equazione integro–differenziale di tipo iperbolico che interviene in un problema di ottimizzazione considerato in [1].

In this paper we investigate an integro-differential equation of hyperbolic type which occurs in an optimization problem considered in [1].

1. INTRODUCTION

It is well known that hyperbolic differential equations with two independent variables can be written in two equivalent forms, called the *first* and the *second canonical form*, by certain classical transformations:

$$\frac{\partial^2 u}{\partial x \, \partial y} = F_1\left(x , y , u , \frac{\partial u}{\partial x} , \frac{\partial u}{\partial y}\right) \tag{1.1}$$

and

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = F_2\left(x , y , u , \frac{\partial u}{\partial x} , \frac{\partial u}{\partial y}\right) \tag{1.2}$$

respectively, where F_1 and F_2 are certain given functions. For each canonical form several well-posed problems have already been investigated. The first canonical form (1.1), for exemple, is particularly suitable for a Darboux-Goursat problem when we use the Riemann method, and the second canonical form (1.2) is convenient for a mixed problem when we use the Fourier method. For a general Darboux-Goursat problem associated with (1.1) we refer to M. Picone [2]. For the Fourier method in connection with the solution of a mixed problem for (1.2) we refer to A. N. Tikhonov and A. A. Samarskii [3].

In the present paper we consider a hyperbolic equation of the form (1.2) where F_2 is a linear functional in u in the following form:

$$\frac{\partial^2 u(x , y)}{\partial x^2} - a^2 \frac{\partial^2 u(x , y)}{\partial y^2} - u(x , y) = f(x , y) + \lambda \iint_{\mathcal{R}} \mathcal{K}(x , y ; \xi , \eta)\, u(\xi , \eta)\, d\xi\, d\eta \tag{1.3}$$

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for $(x, y) \in \mathscr{R}$, subject to the mixed conditions

$$(1.4)\qquad \begin{cases} u(0, x) = \left.\dfrac{\partial u(x, y)}{\partial x}\right|_{x=0} = 0 & (0 \leq y \leq 1) \\ u(x, 0) = u(x, 1) = 0 & (0 \leq x \leq 1), \end{cases}$$

where $\mathscr{R} = \{(x, y) \mid 0 \leq x, y \leq 1\}$, a is a given constant, λ is a real parameter, $f(x, y)$ and $\mathscr{K}(x, y; \xi, \eta)$ are certain given functions on $\mathscr{R}$ and $\mathscr{R} \times \mathscr{R}$ respectively. We assume that the functions $f(x, y)$ and $\mathscr{K}(x, y; \xi, \eta)$ are sufficiently smooth so that their Fourier series with respect to y,

$$(1.5)\qquad f(x, y) = \sum_{n=1}^{\infty} f(x) \sin n\pi y \quad , \quad \mathscr{K}(x, y; \xi, \eta) = \sum_{n=1}^{\infty} \mathscr{K}_n(x; \xi, \eta) \sin n\pi y,$$

are absolutely and uniformly convergent in $\mathscr{R}$ and $\mathscr{R} \times \mathscr{R}$ respectively, where the Fourier coefficients $f(x)$, $\mathscr{K}(x; \xi, \eta)$ are assumed to be continuous for $0 \leq x \leq 1$ and $0 \leq x, \xi, \eta \leq 1$, and the series $\sum_{n=1}^{\infty} n \|f_n\|$ and $\sum_{n=1}^{\infty} n \|\mathscr{K}_n\|$ are convergent, where

$$(1.6)\qquad \|f_n\| = \max_{0 \leq x \leq 1} |f_n(x)| \quad , \quad \|\mathscr{K}_n\| = \max_{0 \leq x, \xi, \eta \leq 1} |\mathscr{K}_n(x; \xi, \eta)|, \qquad n = 1, 2, 3, \cdots.$$

In our next paper we shall investigate a Goursat problem for an integro-differential equation of the form

$$(1.7)\qquad \frac{\partial^2 u}{\partial x\, \partial y} + p(x, y) \frac{\partial u}{\partial x} + q(x, y) \frac{\partial u}{\partial y} + r(x, y)\, x = f(x, y) + \lambda \iint_{\mathscr{R}} \mathscr{K}(x, y; \xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta,$$

where p, q, r, and $\mathscr{K}$ are given functions, and we shall extend the results of M. Picone [2] to the above equation.

The mixed problem (1.3), (1.4) occurs in an optimization problem considered in [1]. In the present note we establish the solution of this problem for small λ.

For simplicity we assume that $|a| > 1/\pi$, although the case $|a| < 1/\pi$ does not present any major difficulty.

2. SOLUTION FOR $\lambda = 0$

For $\lambda = 0$ the integro-differential equation (1.1) reduces to the following hyperbolic equation:

$$(2.1)\qquad \left(\frac{\partial^2}{\partial x^2} - a^2 \frac{\partial^2}{\partial y^2} - 1\right) u(x, y) = f(x, y).$$

Using well known methods of Mathematical Physics we can easily show that the solution of Eq. (2.1) satisfying the conditions (1.4) is of the form

$$u_0(x,y)=\sum_{n=1}^{\infty}\alpha_n(x)\sin n\pi y\,, \tag{2.2}$$

where

$$\alpha_n(x)=\frac{1}{\rho}\int_0^x f(\xi)\sin\rho_n(x-\xi)\,d\xi \quad , \quad \rho_n=\sqrt{n^2\pi^2a^2-1}\,, \tag{2.3}$$

$$n=1,2,3,\cdots.$$

Thus, putting

$$G(x,y;\xi,\eta)=\sum_{n=1}^{\infty}\frac{\sin\rho_n(x-\xi)\sin n\pi y\sin n\pi\eta}{\sqrt{2}\,\rho_n} \tag{2.4}$$

and using the uniform and absolute convergence of the series involved, we find

$$u_0(x,y)=\int_0^x d\xi\int_0^1 G(x,y;\xi,\eta)\,f(\xi,\eta)\,d\eta\,. \tag{2.5}$$

Let us note that

$$\|\alpha_n\|\le\frac{\|f_n\|}{\rho_n}\quad,\quad\|\alpha_n'\|\le\|f_n\|\quad,\quad\|\alpha_n''\|\le(1+\rho_n)\|f_n\|\,, \tag{2.6}$$

$$n=1,2,3,\cdots.$$

Further, in addition to the series (2.2), the series

$$\sum_{n=1}^{\infty}\alpha_n'(x)\sin n\pi y\quad,\quad\sum_{n=1}^{\infty}\alpha_n''(x)\sin n\pi y\quad,\quad-\sum_{n=1}^{\infty}\pi n\alpha_n(x)\cos n\pi y,$$

$$-\sum_{n=1}^{\infty}\pi^2n^2\alpha_n(x)\sin n\pi y$$

are absolutely and uniformly convergent on $\mathscr{R}$ and represent the functions $u(x,y)$, $\frac{\partial u(x,y)}{\partial x}$, $\frac{\partial^2 u(x,y)}{\partial x^2}$, $\frac{\partial u(x,y)}{\partial y}$ and $\frac{\partial^2 u(x,y)}{\partial y^2}$, respectively. Putting $\|W\|=\max_{\mathscr{R}}|W(x,y)|$ for any function $W=W(x,y)$ defined on $\mathscr{R}$, we easily find

$$\|u_0\|\le\sum_{n=1}^{\infty}\frac{\|f_n\|}{\rho_n}\quad,\quad\left\|\frac{\partial u_0}{\partial x}\right\|\le\sum_{n=1}^{\infty}\|f_n\|\quad,\quad\left\|\frac{\partial u_0}{\partial y}\right\|\le\sum_{n=1}^{\infty}n\pi\frac{\|f_n\|}{\rho_n}\,,$$

$$\left\|\frac{\partial^2 u_0}{\partial x^2}\right\|\le\sum_{n=1}^{\infty}(1+\rho_n)\|f_n\|\quad,\quad\left\|\frac{\partial^2 u_0}{\partial y_\eta^2}\right\|\le\sum_{n=1}^{\infty}n^2\pi^2\frac{\|f_n\|}{\rho_n}\,. \tag{2.7}$$

According to the first of the inequalities (2.6), the series $\sum_{n=1}^{\infty} n \, \|\alpha_n\|$ is convergent, since $\rho_n = o(n)$ and the series $\sum_{n=1}^{\infty} n \, \|f_n\|$ is convergent by hypothesis.

We can easily verify that the function

$$F(x, y) = \iint_{\mathcal{R}} \mathcal{K}(x, y; \xi, \eta) \, u_0(\xi, \eta) \, d\xi \, d\eta .$$

like the function $u_0(\xi, \eta)$, satisfies the conditions imposed on the function $f(x, y)$.

3. SOLUTION FOR SMALL λ

The solution of Eq. (1.3) subject to the mixed conditions (14) can be easily established for small λ. To this end, let us seek a solution to (1.3), (1.4) in the form

$$u(x, y) = \sum_{m=0}^{\infty} \lambda^m (x, y) \tag{3.1}$$

with

$$\begin{cases} u_m(0, y) = \dfrac{\partial u_m(0, y)}{\partial x} = 0 & (0 \leq y \leq 1) \\ u_m(x, 0) = u_m(x, 1) = 0 & (0 \leq x \leq 1) \quad , \quad m = 0, 1, 2, \cdots . \end{cases} \tag{3.2}$$

By substitution and comparison we immediately find the recursive system of differential equations

$$\left(\frac{\partial^2}{\partial x^2} - a^2 \frac{\partial^2}{\partial y^2} - 1 \right) u_0(x, y) = f(x, y), \tag{3.3}$$

$$\left(\frac{\partial^2}{\partial x^2} - a^2 \frac{\partial^2}{\partial y^2} - 1 \right) u_m(x, y) = \iint_{\mathcal{R}} \mathcal{K}(x, y; \xi, \eta) \, u_{m-1}(\xi, \eta) \, d\xi \, d\eta , \tag{3.4}$$

$$m = 1, 2, 3, \cdots .$$

Since the solution of Eq. (3.3) subject to the conditions (3.2) is given by formula (2.5), the recursive operations (3.4) continue indefinitely. According to the results of § 2, the solution of Eq. (3.4) which satisfies the conditions (3.2) is of the form

$$u_m(x, y) = \int_0^x d\sigma \int_0^1 G(x, y; \sigma, \tau) \, d\tau \iint_{\mathcal{R}} \mathcal{K}(\sigma, \tau; \xi, \eta) \, u_{m-1}(\xi, \eta) \, d\xi \, d\eta , \tag{3.5}$$

$$m = 1, 2, 3, \cdots .$$

It can be easily verified that each $u_m(x, y)$ satisfies the conditions imposed on the function $f(x, y)$, and we have

$$
(3.6)\quad \begin{cases}
\|x_m\| \leq \left(\sum_{n=1}^{\infty} \frac{\|\mathcal{K}_n\|}{\rho_n}\right)^m \|u_0\|, \\
\left\|\frac{\partial u_m}{\partial x}\right\| \leq \left(\sum_{n=1}^{\infty} \|\mathcal{K}_n\|\right)^m \|u_0\|, \\
\left\|\frac{\partial u_m}{\partial y}\right\| \leq \left(\sum_{n=1}^{\infty} n \frac{\|\mathcal{K}_n\|}{\rho_n}\right)^m \pi^m \|u_0\|, \\
\left\|\frac{\partial^2 u_m}{\partial x^2}\right\| \leq \left(\sum_{n=1}^{\infty} (1 + \rho_n) \|\mathcal{K}_n\|\right)^m \|u_0\|, \\
\left\|\frac{\partial^2 u_m}{\partial y^2}\right\| \leq \left(\sum_{n=1}^{\infty} n^2 \frac{\|\mathcal{K}_n\|}{\rho_n}\right)^m \pi^m \|u_0\|,
\end{cases}
$$

Thus the series (3.1) and its term by term derivatives up to second order are absolutely and uniformly convergent on $\mathcal{R}$ for

$$
(3.7)\qquad |\lambda| < \frac{1}{\sum_{n=1}^{\infty} (1 + \rho_n) \|\mathcal{K}_n\|} .
$$

Thus, we have the following

THEOREM 1. *If* $|a| > 1/\pi$ *and the functions* $f(x, y)$ *and* $\mathcal{K}(x, y; \xi, \eta)$ *are as described in § 1, then the integro-differential equation* (1.3) *admits a unique solution satisfying the conditions* (1.4) *for the values of* λ *satisfying the inequality* (3.7).

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