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On certain transformations for black-holes energetics

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Fisica matematica. — *On certain transformations for black-holes energetics.* Nota di ANNA CURIR (*) e MAURO FRANCAVIGLIA, (**) presentata (***) dal Socio C. AGOSTINELLI.

RIASSUNTO. — È contenuto nell'Introduzione.

1. INTRODUCTION

In the study of the energetics of Kerr black holes with $L \leq m^2$ it has been deduced ([3, 5]) the following relation:

$$m^2 = m_{ir}^2 + L^2/4m_{ir}^2 , \tag{1}$$

which is known under the name of 'fundamental formula of black holes energetics' ([8], p. 540). Besides the parameters m and L characterizing the black hole, it contains the parameter m_{ir}, which is called 'irreducible mass' and it was introduced by Christodoulou and Ruffini ([4, 5]) by means of reversible transformations (see [2], pp. 166–177). These Authors fixed for such transformations the following range:

$$m_{ir}^2 \leq m^2 \leq 2 \cdot m_{ir}^2 , \tag{2}$$

hence restricting to this range also the validity of (1) with $m_{ir} = \text{const.}$

The need of possibly extending the meaning of (1) for $m_{ir} = \text{const.}$ and $m^2 > 2 \cdot m_{ir}^2$, has led us to the present research. In this paper we recognize the necessity of generalizing and clarifying the concept of reversible transformations, by introducing so-called 'isoareal transformations', deepening their relations with the geometry of Kerr solution and in particular with the hypersurfaces $r = r_{\pm}$ known as 'horizons'. Consequently, we propose to re-write the relation (1) in the following more significant form:

$$m^2 = m_{ir}^2 + E_{re}^2 , \tag{3}$$

where E_{re}, which is naturally related to the angular momentum of the black hole, is shown to be deeply related with the inner horizon $r = r_-$. The isoareal transformations are then interpreted as transformations for which either m_{ir} or E_{re} is constant, this two quantities interchanging their role always in an extreme Kerr black hole.

Furthermore, we suggest an interpretation for these isoareal transformations in terms of energy exchanges on the inner horizon, from analogy to the detailed study on the event horizon given by Christodoulou ([3]) for reversible transformations, leaving open the possibility of further studies.

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2. PRELIMINARIES

We recall that, in Boyer and Lindquist coordinates ([1]), the Kerr solution is given by:

$$(4) \qquad ds^2 = (r^2 + a^2 \cos^2 \theta)^2 [\Delta^{-1} dr^2 + d\theta^2] + (r^2 + a^2 \cos^2 \theta)^{-2} \cdot$$
$$\cdot \{\sin^2 \theta \, [a dt - (r^2 + a^2) \, d\varphi]^2 - \Delta \, [dt - a \sin^2 \theta \, d\varphi]^2\} ,$$

where $a = L/m$, m, $L \in \mathbf{R}^+$, and $\Delta(r) = r^2 - 2mr + a^2$. This solution describes an empty space-time external to a rotating and uncharged object, m and L assuming the meaning of its mass and angular momentum.

Whenever $L \leq m^2$, the equation $\Delta(r) = 0$ admits two real roots $r_\pm$. The hypersurfaces $r = r_+$ and $r = r_-$ behave in this case as horizons (see [2, 7]) and they are respectively called event- and inner horizon. Hence, the region $r < r_+$ containing a true singularity (namely, the circle $r = 0$), takes the name of Kerr black hole.

Since in this paper we are dealing with transformations of Kerr black holes under the limitation $L \leq m^2$ (1), we shall agree to represent the set of such black holes as the dashed region $\mathscr{F}$ in fig. 1. Each (continuous) transformation of a Kerr black hole, i.e. of its parameters m and L, will then be represented by a continuous curve in $\mathscr{F}$ (2).

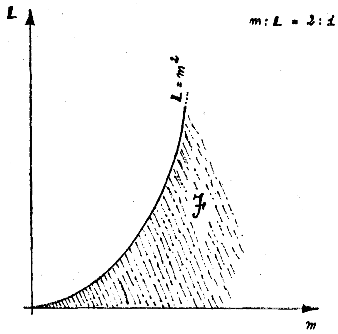

Fig. 1.

Let's now assign a black hole with parameters $(m, L) \in \mathscr{F}$. It is well known ([3], p. 54) that the energy E of a particle on the hypersurface $r = r_0$ is given by the positive root of:

$$(5) \qquad E^2 [r_0^3 + a^2 (r_0 + 2m)] - 4ma \, E p_\varphi - (r_0 - 2m) \, p_\varphi^2 -$$
$$- \Delta(r_0) \, \mu^2 r_0 - \Delta(r_0)^2 \, p_r^2 / r_0 = 0 ,$$

(1) To avoid the appearance of naked singularities ([7], p. 311).

(2) This is of course valid in the limit of assuming matter continuously distributed rather than discrete.

where p_φ = azymuthal momentum, p_r = radial momentum and μ = rest mass of the particle. This energy E has a minimum on the hypersurfaces $r = r_\pm$, since there the discriminant:

$$D = (4\, map_\varphi)^2 + 4\,[r_0^3 + a^2(2\,m + r_0)]\cdot[(r_0 - 2\,m)\,p_\varphi^2 + \\ + \Delta(r_0)\,\mu^2 r_0 + \Delta(r_0)^2\, p_r^2/r_0] \tag{6}$$

is vanishing. The energy E_+ of a particle on the event horizon is then given by:

$$E_+ = \frac{ap_\varphi}{r_+^2 + a^2} \qquad \text{(cfr. [3, 4]).} \tag{7}$$

From integration of the infinitesimal form of (7) [3], Christodoulou ([3]) obtained the following relation:

$$(1 - L^2/m^4)^{1/2} = \frac{2\,\beta}{m^2} - 1\,, \qquad m^2 \leq 2\,\beta\,, \tag{8}$$

where $2\,\beta$ denotes the integration constant. We soon notice the physical meaning of the constant β: in fact, for any fixed $\beta_0 \in \mathbf{R}^+$, $\sqrt{\beta_0}$ is the mass of the unique Schwarzschild black hole ($L = 0$) lying on the curve $L = L(m\,;\beta_0)$ implicitly defined by eqn. (8).

This fact suggested to Christodoulou the position $\beta = m_{ir}^2$, thus implying the limitation $m^2 \leq 2\,m_{ir}^2$; under such an assumption, we get (1) by squaring (8).

Let's then consider the following relation:

$$(1 - L^2/m^4)^{1/2} = 1 - \frac{2\,\beta}{m^2}\,, \qquad m^2 \geq 2\,\beta\,. \tag{9}$$

Since the squaring of (9) leads to the same result obtained by squaring (8), it will be natural to consider (9) as the obvious extension of (8) to values of m^2 greater than $2\,\beta$.

However, we can easily recognize that (9) can be deduced, analogously to (8), from the integration of the infinitesimal form of:

$$E_- = \frac{ap_\varphi}{r_-^2 + a^2}\,, \tag{10}$$

which is now the energy E_- of a particle on the inner horizon. We can so assign to the constant β appearing in (9) the following physical meaning: given the curve $L = L(m\,;\beta_0)$ implicitly defined by (9), $\sqrt{2\,\beta_0}$ is the mass

(3) Namely, by posing $E_+/p_\varphi = dm/dL$.

m_{ext} of the unique extreme Kerr black hole [4] lying on such a curve. It is then quite easy to verify that m_{ext}^2 is equal to the value $2\,m_{ir}^2$ corresponding to the curve belonging to the family (8) which admits endpoint in $(m_{\text{ext}}\,, m_{\text{ext}}^2)$ [5].

Hence, if we fix a constant $\beta \in \mathbf{R}^+$, (8) and (9) are summarized by the following:

$$m^2 = \beta + \frac{L^2}{4\,\beta}\,, \tag{11}$$

where $\sqrt{\beta}$ shall denote the mass of the unique Schwarzschild black hole belonging to the family defined by (11) itself.

Before going on, it will be useful and enlightening, in accordance to the dualism matter $\equiv$ geometry proper to General Relativity, to give a completely geometrical interpretation of (8) and (9) (equiv. of (11)).

Denote by A_+ and A_- the areas of the surfaces ($r = r_\pm\,, t = \text{const.}$; see [6]), namely:

$$A_\pm = 8\,\pi m\,(m \pm \sqrt{m^2 - a^2}\,)\,. \tag{12}$$

We can easily recognize that, along (11), the following relations hold:

$$A_+ = 16\,\pi\beta \qquad \text{when} \quad m^2 \leq 2\,\beta\ ^{(6)}\,; \tag{13 I}$$

$$A_- = 16\,\pi\beta \qquad \text{when} \quad m^2 > 2\,\beta\,; \tag{13 II}$$

namely, along the integral curves of (7) (resp. (10)) the area A_+ (resp. A_-) is constant. This is in perfect agreement with the easily provable fact that (8) constitutes the general solution of the following differential equation:

$$\frac{dA_+}{dm} = \frac{\partial A_+}{\partial m} + \frac{\partial A_+}{\partial a}\cdot\frac{da}{dm} = 0\,, \tag{14 I}$$

while (9) is the general solution of:

$$\frac{dA_-}{dm} = 0\,. \tag{14 II}$$

3. ISOAREAL TRANSFORMATIONS AND THEIR PHYSICAL MEANING

The purpose of this Sect. is to study those transformations of a Kerr black hole which are governed by eqn. (11), among which the 'reversible transformations' studied by Christodoulou and Ruffini ([3, 4, 5]) constitute

(4) We call extreme Kerr black holes those with $L = m^2$.

(5) In § 3 it will be clear that such a curve exists and it is unique.

(6) Cf. p. 96 in [3], there posing $\beta = m_{ir}^2$.

the particular case $m^2 \leq 2\beta$. In virtue of the above considerations, we shall call them with the name of 'isoareal transformations'.

The relation (11) defines, for $m, L \geq 0$, a family of positive arcs of hyperbolas. It is easily verified that the parabola $L = m^2$ delimiting the region $\mathcal{F}$ of physical interest is the enveloping curve of the family (11), the tangency point being, for any $\beta \in \mathbf{R}^+$, the point $P_{e\beta} = (\sqrt{2\beta}, 2\beta)$. Hence, the hyperbolas (11) always lye in the region $\mathcal{F}$, i.e. the only region where A_+ and A_- have a meaning, filling it completely. More precisely, we easily recognize that for any $P_0 = (m_0, L_0) \in \mathcal{F}$ we can draw the two hyperbolas:

$$i_{0\pm} \equiv m^2 = \beta_\pm + \frac{L^2}{4\beta_\pm} \tag{15}$$

where $\beta_\pm = \frac{1}{2}(m_0^2 \pm \sqrt{m_0^4 - L_0^2})$. We notice that β_+ and β_- are identified (besides the numerical factor 16π) with the areas $A_\pm^{(0)}$ corresponding to the black hole P_0, according to relations (13).

The hyperbolas $i_{0\pm}$ intersect the m-axis in two Schwarzschild black holes, namely $P_{s\pm} = (\sqrt{\beta_\pm}, 0)$, such that the mass of P_{s-} is always lesser than the mass of P_{s+}. The extreme black hole P_{e-} lying on i_{0-} is always on the left of P_0, while P_{e+} (on i_{0+}) is always on the right of P_0 (see fig. 2).

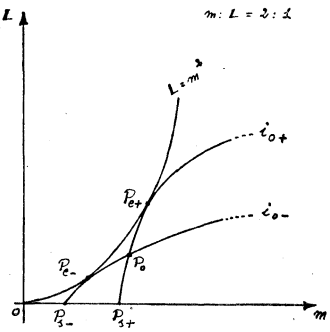

Fig. 2.

According to Sect. 2, the mass $\sqrt{\beta_+}$ has hence the meaning of 'irreducible mass'; it is indeed the mass of P_{s+} and, in conformity to (13 I), A_+ is constant along i_{0+}, being everywhere equal to $16\pi\beta_+$. Which is the meaning that we can assign to the mass $\sqrt{\beta_-}$ of P_{s-}? Let's notice that it is conserved, together with A_-, along the hyperbola i_{0-}. According to Sect. 2, $\sqrt{2\beta_-}$ is the mass of the extreme black hole P_{e-}; hence, $\beta_- = \frac{1}{2} L_{ext}$, where L_{ext} denotes the angular momentum of P_{e-}. This circumstance suggests us to interpret β_- as the rotational part of the mass-energy of P_0.

This last idea is furthermore supported by the following simple but clear fact: the 'geometrical' difference between a Schwarzschild black hole and a Kerr black hole is manifestly represented by the appearance of the inner horizon $r = r_-$. In fact, for $L \cong 0$ we easily get $r_+ \cong 2m$ (Schwarzschild radius) and, by a power expansion, $r_- \cong \frac{1}{2}(L^2/m^3) \cong 0$, thus emphasizing that the presence of the angular momentum L is strictly related to the existence of r_-.

We then notice that the areas A_+ and A_- of any black hole $(m, L) \in \mathscr{F}$ satisfy the following:

$$A_+ \cdot A_- = 64\,\pi^2 L^2 . \tag{16}$$

Along each hyperbola (11) A_+ is constant (precisely $A_+ = 16\,\pi\beta$) till the point $P_{e\beta}$, while $A_-(m)$ is monotonically increasing with m from the value 0 to the maximum value $A_- = 16\,\pi\beta$ (7). The transformations involved are 'reversible' in the thermodynamical sense of Carter ([2]), since the variation δA_+ is vanishing; this justify the name 'irreducible mass' given to $\sqrt{\beta}$ in such part of the hyperbola (11): it is in fact identified with the area A_+ which, by virtue of a theorem due to Hawking ([6]), can never decrease in any possible transformation of a black hole. Along the part of the hyperbola lying after $P_{e\beta}$, on the contrary, the area A_- is constant, retaining its previous maximum value $16\,\pi\beta$; $A_+(m)$ is then, by (16), increasing with L like $16\,\pi\beta^{-1} L^2$.

It is interesting to notice that these last transformations are no longer reversible: in fact, Hawking's theorem $\delta A_+ \geq 0$ and relation (16) explicitly impose them to happen only in the sense of increasing L.

In any case, they can be interpreted (according to the derivation of (9) from (10)) as energy exchanges on the inner horizon $r = r_-$, from analogy to the elegant interpretation given by Christodoulou to the transformations governed by (8) (see [3, 4]). This fact emphasizes once more the non-reversible character of isoareal transformations with $A_- =$ const.: they indeed assume, in such an interpretation, the meaning of 'internal transformations', hence no longer giving raise to the possibility of an energy extraction, which is on the contrary possible for reversible ones (see [3], p. 99). Nevertheless, these internal isoareal transformations are physically observable, although happening inside the event horizon itself; they indeed require variation of the area A_+, which is of course 'observable' from outside the black hole.

The above considerations have thus suggested us the following generalization of the concepts exposed in [4], a generalization which furthermore makes, in our opinion, clearer and more symmetric the study of (11). Given a black hole $P = (m, L) \in \mathscr{F}$ we define its 'irreducible mass' to be the quantity $A_+/16\,\pi$; we call 'extreme rotational energy' the quantity E_{re} defined by:

$$E_{re}^2 = L^2/4\,m_{ir}^2 = A_-/16\,\pi \quad ^{(8)}. \tag{17}$$

(7) In $P_{e\beta}$ we have $r_+ = r_-$, $P_{e\beta}$ being extreme.

(8) The last equality being valid by virtue of (10).

Under these assumptions the fundamental formula of black hole energetics takes the following form:

$$m^2 = m_{ir}^2 + E_{re}^2 . \tag{18}$$

The two quantities m_{ir} and E_{re} have the following meaning: m_{ir} is the mass of the unique Schwarzschild black hole which can be obtained from P by means of reversible transformations (i.e. along the hyperbola i_+ through P); $2 E_{re}$ is the angular momentum of the unique extreme black hole from which we can obtain P by means of internal isoareal transformations (i.e. along i_- through P). The isoareal transformations (11) are characterized by the constancy of one and only one of the two quantities above, which interchange their roles in the tangency point $P_{e\beta}$. The details are left to the reader.

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