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Awdhesh Kumar

On a special recurrent Finsler space of second order

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Geometria differenziale. — *On a special recurrent Finsler space of second order.* Nota di AWDHESH KUMAR, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Il presente lavoro si ricollega alla ricerca [4] dello stesso Autore, dedicata allo studio dei moti affini negli spazi ricorrenti di Finsler del secondo ordine. Qui si tratta di tali spazi, sotto le condizioni (2.1), (2.2) e (2.3).

INTRODUCTION

Let us consider an n-dimensional affinely connecting Finsler space F_n [1] [(1)] equipped with $2n$ line elements and a fundamental metric function $F(x, \dot{x})$ which is positively homogeneous of degree one in its directional arguments. The fundamental metric tensor of the space is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \tfrac{1}{2}\, \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \qquad \dot{\partial}_i \equiv \partial/\dot{x}^i \tag{1.1}$$

and is symmetric in its lower indices.

Let us further consider a mixed tensor field $T^i_j(x, \dot{x})$ which depends both upon positional and directional arguments. The covariant derivative of $T^i_j(x, \dot{x})$ with respect to x^k in the sense of Cartan is given by

$$T^i_{j|k} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_k + T^s_j \Gamma^{*i}_{sk} - T^i_s \Gamma^{*s}_{jk}, \tag{1.2}$$

where $\Gamma^{*i}_{hk}(x, \dot{x})$ are connection coefficients and are also symmetric in their lower indices j and k.

The commutation formula involving the above covariant derivative for the tensor field $T^i_j(x, \dot{x})$ is given by

$$2\, T^i_{j|[hk]} = -\dot{\partial}_r T^i_j K^r_{hk} + T^s_j K^i_{shk} - T^i_s K^s_{jhk} \text{ [(2)]}, \tag{1.3}$$

where

$$K^i_{hjk}(x, \dot{x}) \overset{\text{def}}{=} 2\, \{\partial_{[k} \Gamma^{*i}_{j]h} - \dot{\partial}_r \Gamma^{*i}_{h[j} G^r_{k]} + \Gamma^{*r}_{h[j} \Gamma^{*i}_{k]r}\} \tag{1.4}$$

is called Cartan's curvature tensor field and satisfies the following identities [1]:

$$K^i_{hjk} + K^i_{jkh} + K^i_{khj} = 0 \tag{1.5}$$

and

$$a)\ \ K^i_{hjk} = -K^i_{hkj} \qquad \text{and} \qquad b)\ \ K^i_{hji} = K_{hj}. \tag{1.6}$$

(*) Nella seduta del 13 novembre 1976.

(1) The numbers in square brackets refer to the References given at the end of the paper.

(2) $2\, A_{[hk]} = A_{hk} - A_{kh}$.

2. RECURRENT FINSLER SPACE OF SECOND ORDER

DEFINITION (2.1). An n-dimensional affinely connected F_n is called recurrent Finsler space of the second order if its curvature tensor field $K^i_{hjk}(x, \dot{x})$ satisfies the relation:

$$K^i_{hjk|sm} = \alpha_{sm} K^i_{hjk}, \tag{2.1}$$

where $\alpha_{sm}(x, \dot{x})$ is a non-zero symmetric tensor field.

In what follows, we shall assume the following two conditions:

$$v^i_{|h} = \varepsilon_h v^i \tag{2.2}$$

and

$$K_{hj} = \varepsilon_h \eta_j, \tag{2.3}$$

where η_j is a suitable covariant vector and $K_{hj}(x, \dot{x})$ is the so called Ricci tensor given by (1.6 b). In the following, we shall study the fundamental properties of the space under the conditions (2.1), (2.2) and (2.3).

Taking the covariant derivative of (2.2) with respect to x^s and using the same equation (2.2), we have

$$v^i_{|hs} = v^i (\varepsilon_{h|s} + \varepsilon_h \varepsilon_s). \tag{2.4}$$

By virtue of the commutation formula (1.3), commutating (2.4) with respect to the indices h and s, we get

$$K^i_{mhs} v^m = \tilde{\Omega}_{hs} v^i, \tag{2.5}$$

where

$$\tilde{\Omega}_{hs} \overset{\text{def}}{=} (\varepsilon_{h|s} - \varepsilon_{s|h}). \tag{2.6}$$

In view of the equation (1.6 b) contracting the definition (2.1) with respect to the indices i and k, we obtain

$$K_{hj|sm} = \alpha_{sm} K_{hj}. \tag{2.7}$$

Introducing (2.3) into the left hand side of the above equation, we have

$$\varepsilon_{h|s} \eta_{j|m} + \eta_j \varepsilon_{h|sm} + \varepsilon_{h|m} \eta_{j|s} + \varepsilon_h \eta_{j|sm} = \alpha_{sm} \varepsilon_h \eta_j. \tag{2.8}$$

Making a commutator on the indices s and m from the last formula, we get

$$-\eta_j \varepsilon_r K^r_{hsm} - \varepsilon_h \eta_r K^r_{jsm} = Q_{sm} \varepsilon_h \eta_j, \tag{2.9}$$

where

$$Q_{sm} \overset{\text{def}}{=} \alpha_{sm} - \alpha_{ms}. \tag{2.10}$$

Multiplying (2.9) by v^h and summing over h, we obtain

$$\varepsilon\,(\tilde{\Omega}_{sm}\,\eta_j + \eta_r\,K^r_{jsm} + \eta_j\,Q_{sm}) = 0\,, \tag{2.11}$$

where we put

$$\varepsilon \overset{\text{def}}{=} \varepsilon_h\,v^h\,. \tag{2.12}$$

In this way, we have to discuss here the next two cases:

$$a)\quad \tilde{\Omega}_{sm}\,\eta_j + \eta_r\,K^r_{jsm} + \eta_j\,Q_{sm} = 0 \qquad \text{and} \qquad b)\quad \varepsilon = 0\,. \tag{2.13}$$

2. The case $\eta = 0$

By putting

$$\eta = \eta_h\,v^h \tag{3.1}$$

condition (2.13 a) yields this case. We shall show this fact. In wiew of the above equation transvecting (2.13 a) by v^j and summing over j, we get

$$\eta\,(2\,\tilde{\Omega}_{sm} + Q_{sm}) = 0. \tag{3.2}$$

Consequently for the present case (2.13 a) we have to consider two cases:

$$a)\quad \eta = 0 \qquad \text{and} \qquad b)\quad 2\,\tilde{\Omega}_{sm} + Q_{sm} = 0\,. \tag{3.3}$$

Transvecting the latter case by v^i and making use of (2.5), we obtain

$$Q_{sm}\,v^i + 2\,K^i_{rsm}\,v^r = 0. \tag{3.4}$$

By virtue of the equation (1.6 b) contracting the last relation with respect to the indices i and m, we have

$$Q_{sm}\,v^m + 2\,K_{rs}\,v^r = 0\,. \tag{3.5}$$

Introducing the equations (2.3) and (2.12) into the above relation we get

$$Q_{sm}\,v^m + 2\,\varepsilon\eta_s = 0\,. \tag{3.6}$$

Multiplying this equation by v^s and noting (2.3), (2.12) and (3.1) we obtain

$$Q_{sm}\,v^s\,v^m + 2\,\varepsilon\eta = 0 \tag{3.7}$$

or

$$\varepsilon\eta = 0\,. \tag{3.8}$$

In this way, we have $\eta = 0$, i.e. the second case (3.3 b) means first case (3.3 a) and (2.13 a) may be replaced by $\eta = 0$. By this reason there exists only one case $\eta = 0$.

By virtue of the commutation formula (1.3), commutating the indices s and m of the definition (2.1), we get

$$(3.9) \qquad Q_{sm} K^i_{hjk} = - \dot\partial_r K^i_{hjk} K^r_{sm} + K^r_{hjk} K^i_{rsm} - K^i_{rjk} K^r_{hsm} - - K^i_{hrk} K^r_{jsm} - K^i_{hjr} K^r_{ksm} ,$$

where we have used the equation (2.10) also.

Contracting the last relation with respect to the indices i and k we have

$$(3.10) \qquad Q_{sm} K_{hj} = - K_{hr} K^r_{ism} - K_{rj} K^r_{hsm} ,$$

where we have made use of (1.6).

In view of the equations (2.3), (2.5) and (2.12) transvecting the last equation by v^h and summing over h, we get

$$(3.11) \qquad \eta_r K^r_{jsm} = - \eta_j (Q_{sm} + \tilde\Omega_{sm}) ,$$

where we have used the fact that $\varepsilon \neq 0$.

Introducing (3.11) into the left hand side of the equation (2.9), we obtain

$$(3.12) \qquad \eta_j (\varepsilon_r K^r_{hsm} - \varepsilon_h \tilde\Omega_{sm}) = 0 .$$

In this way, there occur the following two cases to be discussed:

$$(3.13) \qquad a) \quad \eta_j = 0 \qquad \text{and} \qquad b) \quad \varepsilon_r K^r_{hsm} = \varepsilon_h \tilde\Omega_{sm} .$$

Introducing the Bianchi identity (1.5) for the curvature tensor $K^r_{hsm}(x, \dot x)$ in the case of (3.13 b), we have

$$(3.14) \qquad \varepsilon_h \tilde\Omega_{sm} + \varepsilon_s \tilde\Omega_{mh} + \varepsilon_m \tilde\Omega_{hs} = 0 .$$

Transvecting the last identity by v^h and summing over h, we get

$$(3.15) \qquad \tilde\Omega_{sm} = \eta_s \varepsilon_m - \eta_m \varepsilon_s ,$$

where we have used $\tilde\Omega_{hm} = - \tilde\Omega_{mh}$ and $\tilde\Omega_{mh} v^h = - K_{mh} v^h$.

In view of definitions (2.3) and (2.6), we can write down an interesting formula:

$$(3.16) \qquad K_{sm} - K_{ms} = \varepsilon_{m|s} - \varepsilon_{s|m} .$$

By virtue of the equations (2.5) and (3.13 b), we have

$$(3.17) \qquad \varepsilon_r K^r_{hms} v^i = \varepsilon_h K^i_{rsm} v^r$$

and this means

$$(3.18) \qquad K^r_{hsm} v^i_{|r} - K^i_{rsm} v^r_{|h} = 0 .$$

That is to say, we actually have

$$(v^i_{|h})_{|sm} - (v^i_{|h})_{|ms} = 0 . \tag{3.19}$$

Consequently we can imagine the existence of a gradient vector and we are able to put

$$v^i_{|sm} = \sigma_m v^i_{|s} . \tag{3.20}$$

In view of equations (2.2) and (3.20), we can conclude that

$$\varepsilon_{h|s} + \varepsilon_h \varepsilon_s = \sigma_s \varepsilon_h . \tag{3.21}$$

Transvecting the above equation by v^h and using the relation (2.12), we have

$$\begin{aligned} \varepsilon\sigma_s &= \varepsilon_{h|s} v^h + \varepsilon\varepsilon_s = (\varepsilon_h v^h)_{|s} - \varepsilon_h v^h_{|s} + \varepsilon\varepsilon_s \\ &= \varepsilon_{|s} - \varepsilon_h \varepsilon_s v^s + \varepsilon\varepsilon_s = \varepsilon_{|s} - \varepsilon\varepsilon_s + \varepsilon\varepsilon_s \\ &= \varepsilon_{|s} . \end{aligned} \tag{3.22}$$

In this way, the existence of σ_s is examined and we have here a characteristic condition on $v^i_{|h}$:

$$v^i_{|hs} = \sigma_s v^i_{|h} \quad , \quad \sigma_s = \sigma_{|s}/\sigma \tag{3.23}$$

On the other hand, in case of (3.13 a), from (2.3), we have

$$K_{hj} = 0 . \tag{3.24}$$

Summarizing all the considerations above, we can state the following:

THEOREM (3.1). *In an n-dimensional second order recurrent Finsler space admitting a contravariant vector v^i characterized by* (2.2) *and having a disjoint Ricci tensor of the form* (2.3) *there exists a case in which $\eta_m v^m = 0$. In this case, if $\eta_m = 0$, then we have the vanishing of Ricci tensor K_{hj} and if $\eta_j \neq 0$, we have* (3.16). *The mixed tensor $v^i_{|h}$ itself is a recurrent one characterized by* (3.23) *for a definite gradient vector $\sigma_s = \sigma_{|s}/\sigma$.*

4. THE CASE OF $\varepsilon = 0$

Let us consider on the case of (2.13 b). Then using an analogous method as in η 3 transvecting (3.10) by v^j, we have

$$\varepsilon_r K^r_{hsm} = -\varepsilon_h (Q_{sm} + \tilde{\Omega}_{sm}) . \tag{4.1}$$

Substituting (4.1) into the left hand side of (2.9), we get

$$\varepsilon_h (\eta_r K^r_{jsm} - \eta_j \tilde{\Omega}_{sm}) = 0 . \tag{4.2}$$

So, we have here two cases to be discussed. They are

$$(4.3)\qquad a)\ \varepsilon_h = 0 \qquad \text{and} \qquad b)\ \eta_r\, K^r_{jsm} = \eta_j\, \tilde{\Omega}_{sm}\,.$$

Case (4.3a) yields the one of $v^i_{|h} = 0$ and $Q_{hj} = 0$. Case (4.3b) may be treated as follows. We can find

$$(4.4)\qquad \eta_h\, \tilde{\Omega}_{sm} + \eta_s\, \tilde{\Omega}_{mh} + \eta_m\, \tilde{\Omega}_{hs} = 0\,.$$

We also have

$$(4.5)\qquad \tilde{\Omega}_{sm}\, v^s = -K_{sm}\, v^s = -\varepsilon_s\, \eta_m\, v^s = -\varepsilon\eta_m = 0\,.$$

In view of the above equation transvecting (4.4) by v^h we get

$$(4.6)\qquad \eta_l \tilde{\Omega}_{ms} = 0\,,$$

where we have also used (3.1). From (2.5) by contraction on the indices i and h, we obtain

$$(4.7)\qquad \tilde{\Omega}_{ms} = 0$$

or

$$(4.8)\qquad \varepsilon_{m|s} = \varepsilon_{s|m}\,.$$

Thus, we can state here the next

THEOREM (4.1). *When* $\varepsilon = 0$ *in our space, there exists two cases. One of them is a case in which* $v^i_{|h} = 0$ *is satisfied and* $K_{hj} = 0$ *and the other is the case of* (4.8).

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REFERENCES

[1] H. RUND (1959) – *The differential geometry of Finsler space*, Springer Verlag.
[2] K. TAKANO (1971) – *On a space with bi-recurrent curvature*, « Tensor, N. S. », *22*, 329–332.
[3] K. TAKANO (1973) – *On a special bi-recurrent space*, « Tensor, N. S. », *27*, 240–242.
[4] A. KUMAR – *Some theorems on affine motion in a recurrent Finsler space*, « Indian Jour. of Pure and Applied Math. » (to appear).