Bang-yen Chen

Value-distributions of Some Deformation Invariants for Bochner-Kaehler Manifolds


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_5_428_0>

RIASSUNTO. — Vengono date varie informazioni concernenti la distribuzione dei valori di due degli invarianti di deformazione di una varietà di Bochner–Kaehler (§ 1, Theor. 1), facendone poi varie applicazioni (§ 4).

§ 1. STATEMENT OF RESULTS

Let $M$ be a compact $n$-dimensional Kaehler manifold with Kaehler metric $g$. Let $\omega$ be the cohomology class represented by the fundamental 2-form $\Phi$ and $c_i$ the $i$-th Chern class of $M$. By Stokes' theorem it is clear that the cohomology classes $\omega^{n-k_i} c_i^{k_i}, 0 \leq k_i \leq n$, are invariant under the Kaehlerian deformations of $g$ which preserve the fundamental class. In the following we shall also denote by $\omega^{n-k_i} c_i^{k_i}$ the real number obtained from $\omega^{n-k_i} c_i^{k_i}$ by taking its value on the fundamental cycle of $M$.

The tensor field introduced by S. Bochner [1] is considered as a complex version of the Weyl conformal curvature tensor and is called the Bochner curvature tensor (definition will be given in § 2). A Kaehler metric is called a Bochner-Kaehler metric if its Bochner curvature tensor vanishes. A complex manifold with a Bochner-Kaehler metric is called a Bochner-Kaehler manifold.

The main purpose of this paper is to prove Theorem 1 which gives some informations about the value-distributions of the deformation invariants $\omega^{n-2} c_1^2$ and $\omega^{n-2} c_2$. Some applications will be given in the last section.

THEOREM 1. Let $M$ be a compact $n$-dimensional Bochner-Kaehler manifold. Then

(a) $\omega^{n-2} c_2 \leq \min \left\{ \frac{n}{2(n+1)} \omega^{n-2} c_1^2, \frac{n}{n+2} \omega^{n-2} c_1^2 \right\}$.

(b) $\omega^{n-2} c_2 = \frac{n}{2(n+1)} \omega^{n-2} c_1^2$ if and only if $\omega^{n-2} c_1^2 \geq 0$ and $M$ is a complex space form.

(c) $\omega^{n-2} c_2 = \frac{n}{n+2} \omega^{n-2} c_1^2 < \frac{n}{2(n+1)} \omega^{n-2} c_1^2$ if and only if $\omega^{n-2} c_1^2 < 0$,

$n$ is even and $M$ is a locally product space of two $(n/2)$-dimensional complex space forms of constant holomorphic sectional curvatures $H (> 0)$ and $-H$.

(*) Partially supported by National Science Foundation under Grant MCS 76-06318.
(**) Nella seduta del 13 novembre 1976.
(1) We shall only consider Kaehler manifolds of complex dimension $n \leq 2$. 
§ 2. Preliminaries

Let $M$ be a compact $n$-dimensional Kaehler manifold. Let $\theta^1, \cdots, \theta^n$ be a local field of unitary coframes. Then the Kaehler metric is written as $g = \sum (\theta^i \otimes \overline{\theta}^i + \overline{\theta}^i \otimes \theta^i)$ and the fundamental 2-form is given by
\[
\Phi = \frac{1}{2} \sum \theta^i \wedge \overline{\theta}^i.
\]
Let $\Omega^i_j = \sum R^i_{jk} \theta^k \wedge \overline{\theta}^l$ be the curvature form of $M$. Then the curvature tensor of $M$ is the tensor field with local components $R^i_{jk} jk$, which will be denoted by $R$. The Ricci tensor $S$ and the scalar curvature $\rho$ are given by
\[
S = \sum (R^i_{ij} \theta^i \otimes \overline{\theta}^j + R^i_{ji} \overline{\theta}^i \otimes \theta^j),
\]
\[
\rho = 2 \sum R^i_{ii},
\]
where $R^i_{ij} = 2 \Sigma R^k_{ij} k$. The Bochner curvature tensor $B$ is a tensor field with local components
\[
B^i_{jk} = R^i_{jk} - \frac{1}{2(n + 2)} (R_{ik} \delta_{jl} + R_{ij} \delta_{kl} + \delta_{ik} R_{jl} + \delta_{ij} R_{kl})
\]
\[
+ \frac{\rho}{4(n + 1)(n + 2)} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}).
\]
We denote by $\|R\|, \|S\|$ and $\|B\|$ the length of the curvature tensor, the Ricci tensor and the Bochner curvature tensor, respectively, so that $\|R\|^2 = 16 \Sigma R^i_{jk} R^k_{ij}, \|S\|^2 = 2 \Sigma R^i_{ij} R^j_{ji}$, and $\|B\|^2 = 16 \Sigma B^i_{jk} B^k_{ij}$. It is easily seen that
\[
\|B\|^2 = \|R\|^2 - \frac{8}{n + 2} \|S\|^2 + \frac{2}{(n + 1)(n + 2)} \rho^2.
\]
We have the following

**Lemma [3].** Let $M$ be an $n$-dimensional Kaehler manifold. Then
\[
\frac{n(n + 1)}{2} \|R\|^2 \geq 2 n \|S\|^2 \geq \rho^2.
\]
The first equality holds if and only if $M$ is a complex space form and the second equality holds if and only if $M$ is Einsteinian.

§ 3. Proof of Theorem 1

Let $M$ be a compact $n$-dimensional Bochner-Kaehler manifold. If we define a closed 2 $k$-form $\gamma_k$ by
\[
\gamma_k = \frac{(-1)^k}{(2 \pi)^{\frac{k + 1}{2}}} \sum \delta^{i_1 \cdots i_k} j_1 \cdots j_k \Omega_{i_1}^j \wedge \cdots \wedge \Omega_{i_k}^j,
\]
then the $k$-th Chern class $c_k$ of $M$ is represented by $\gamma_k$. In particular, $c_1$ and $c_2$ are represented by

$$\gamma_1 = \frac{1}{2\pi i} \sum \Omega_i^1,$$

and

$$\gamma_2 = -\frac{1}{8\pi^2} \sum (\Omega_i^1 \wedge \Omega_j^2 - \Omega_i^1 \wedge \Omega_j^1),$$

respectively. Thus, we see that $\omega^{n-2} c_1^2$ and $\omega^{n-2} c_2$ are represented by

$$\Phi^{n-2} \gamma_1^2$$

and

$$\Phi^{n-2} \gamma_2,$$

respectively. Hence we have

$$\omega^{n-2} c_1^2 = \int_M (\Phi^{n-2} \gamma_1^2) * 1,$$

and

$$\omega^{n-2} c_2 = \int_M (\Phi^{n-2} \gamma_2) * 1,$$

where * denotes the Hodge star operator. Thus by straight-forward computation we get

(2) $$\omega^{n-2} c_1^2 = \frac{(n-2)!}{16\pi^2} \int_M (\rho^2 - 2 \| S \|)^2 * 1,$$

(3) $$\omega^{n-2} c_2 = \frac{(n-2)!}{32\pi^2} \int_M (\rho^2 - 4 \| S \| + \| R \|)^2 * 1.$$

By substituting (1) into (3) we get

(4) $$\omega^{n-2} c_2 = \frac{(n-2)!}{32\pi^2} \int_M \left[ \frac{n(n+3)}{(n+1)(n+2)} \rho^2 - \frac{4n}{n+2} \| S \|^2 \right] * 1.$$

From (2) and (4) we have

$$\omega^{n-2} c_2 = \frac{n}{2(n+1)} \omega^{n-1} c_1^2 = \frac{n}{32(n+1)(n+2)} \int_M (\rho^2 - 2n \| S \|^2) * 1.$$

Thus, by using Lemma, we get

(5) $$\omega^{n-2} c_2 \leq \frac{n}{2(n+1)} \omega^{n-2} c_1^2.$$

On the other hand, by using (2) and (4) again we have

$$\omega^{n-2} c_2 = \frac{n}{n+2} \omega^{n-2} c_1 = \frac{-n!}{32(n+1)(n+2)\pi^2} \int_M \rho^2 * 1 \leq 0,$$
which implies

\[ \omega^{n-2} c_2 \leq \frac{n}{n+2} \omega^{n-2} c_1^2. \]

Since \( \frac{n}{2(n+1)} \leq \frac{n}{n+2} \) for \( n \geq 2 \), (5) and (6) show that \( \omega^{n-2} c_2 \leq \frac{n}{2(n+1)} \omega^{n-2} c_2 \) if \( \omega^{n-2} c_1^2 \geq 0 \) and \( \omega^{n-2} c_2 \leq \frac{n}{n+2} \omega^{n-2} c_1^2 \) if \( \omega^{n-2} c_1^2 < 0 \).

The remaining part of the theorem follows immediately from Theorem 3 of [3]. (Q.E.D.)

§ 4. Applications

A Kaehler manifold \( M \) is said to be cohomologically Einsteinian if \( c_1 = a \omega \) for some constant \( a \). Since \( \omega^n = [\Phi^n] \) and \( \Phi^n = n + 1 \), \( \omega^{n-2} c_1^2 \geq 0 \) for any cohomologically Einstein-Kaehler manifold. Thus Theorem 1 implies immediately the following

**Corollary 1.** Let \( M \) be a compact \( n \)-dimensional cohomologically Einstein Bochner-Kaehler manifold. If \( \omega^{n-2} c_2 \geq \frac{n}{2(n+1)} \omega^{n-2} c_1^2 \), then \( M \) is a complex space form.

Corollary 1 generalizes Theorem 1 of [4].

Combining Corollary 1 with Theorem 1 of [3] we have the following

**Corollary 2** [5]. Let \( M \) be a compact Einstein Bochner-Kaehler manifold. Then \( M \) is a complex space form.

Let \( \tau \) denote the Hirzebruch signature of a Kaehler surface \( M \). Then we have \( \tau = 1/3 (c_1^2 - 2 c_2) \). Thus from Theorem 1 (a) we have the following

**Corollary 3** [2]. Let \( M \) be a compact Bochner-Kaehler surface. Then \( \tau \geq 0 \).

**References**