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Value-distributions of Some Deformation Invariants for Bochner-Kaehler Manifolds

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Geometria differenziale. — *Value-distributions of Some Deformation Invariants for Bochner-Kaehler Manifolds* (*). Nota di BANG-YEN CHEN, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Vengono date varie informazioni concernenti la distribuzione dei valori di due degli invarianti di deformazione di una varietà di Bochner-Kaehler (§ 1, Theor. 1), facendone poi varie applicazioni (§ 4).

§ 1. STATEMENT OF RESULTS

Let M be a compact n-dimensional [1] Kaehler manifold with Kaehler metric g. Let ω be the cohomology class represented by the fundamental 2-form Φ anc c_i the i-th Chern class of M. By Stokes' theorem it is clear that the cohomology classes $\omega^{n-ki} c_i^k$, $0 \leqq ki \leqq n$, are invariant under the Kaehlerian deformations of g which preserve the fundamental class. In the following we shall also denote by $\omega^{n-ki} c_i^k$ the real number obtained from $\omega^{n-ki} c_i^k$ by taking its value on the fundamental cycle of M.

The tensor field introduced by S. Bochner [1] is considered as a complex version of the Weyl conformal curvature tensor and is called the Bochner curvature tensor (definition will be given in § 2). A Kaehler metric is called a Bochner-Kaehler metric if its Bochner curvature tensor vanishes. A complex manifold with a Bochner-Kaehler metric is called a *Bochner-Kaehler manifold*.

The main purpose of this paper is to prove Theorem 1 which gives some informations about the value-distributions of the deformation invariants $\omega^{n-2} c_1^2$ and $\omega^{n-2} c_2$. Some applications will be given in the last section.

THEOREM 1. *Let* M *be a compact n-dimensional Bochner-Kaehler manifold. Then*

(a) $$\omega^{n-2} c_2 \leqq \min \left\{ \frac{n}{2(n+1)} \omega^{n-2} c_1^2 \,,\, \frac{n}{n+2} \omega^{n-2} c_1^2 \right\}.$$

(b) $\omega^{n-2} c_2 = \dfrac{n}{2(n+1)} \omega^{n-2} c_1^2$ *if and only if* $\omega^{n-2} c_1^2 \geqq 0$ *and* M *is a complex space form.*

(c) $\omega^{n-2} c_2 = \dfrac{n}{n+2} \omega^{n-2} c_1^2 < \dfrac{n}{2(n+1)} \omega^{n-2} c_1^2$ *if and only if* $\omega^{n-2} c_1^2 < 0$, *n is even and* M *is a locally product space of two* $(n/2)$*-dimensional complex space forms of constant holomorphic sectional curvatures* H (>0) *and* $-$H.

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(1) We shall only consider Kaehler manifolds of complex dimension $n \leqq 2$.

§ 2. PRELIMINARIES

Let M be a compact n-dimensional Kaehler manifold. Let $\theta^1, \cdots, \theta^n$ be a local field of unitary coframes. Then the Kaehler metric is written as $g = \Sigma(\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i)$ and the fundamental 2-form is given by $\Phi = \frac{\sqrt{-1}}{2} \Sigma \theta^i \wedge \bar{\theta}^i$. Let $\Omega^i_j = \Sigma R^i_{jk\bar{l}} \theta^k \wedge \bar{\theta}^l$ be the curvature form of M. Then the curvature tensor of M is the tensor field with local components $R^i_{jk\bar{l}}$, which will be denoted by R. The Ricci tensor S and the scalar curvature ρ are given by

$$S = \Sigma(R_{i\bar{j}}\, \theta^i \otimes \bar{\theta}^j + \bar{R}_{i\bar{j}}\, \bar{\theta}^i \otimes \theta^j),$$

$$\rho = 2\Sigma R_{i\bar{i}},$$

where $R_{i\bar{j}} = 2\Sigma R^k_{ik\bar{j}}$. The Bochner curvature tensor B is a tensor field with local components

$$B^i_{jk\bar{l}} = R^i_{jk\bar{l}} - \frac{1}{2(n+2)}(R_{\bar{i}k}\,\delta_{jl} + R_{\bar{i}j}\,\delta_{kl} + \delta_{ik}\,R_{j\bar{l}} + \delta_{ij}\,R_{k\bar{l}})$$

$$+ \frac{\rho}{4(n+1)(n+2)}(\delta_{ik}\,\delta_{jl} + \delta_{ij}\,\delta_{kl}).$$

We denote by $\| R \|$, $\| S \|$ and $\| B \|$ the length of the curvature tensor, the Ricci tensor and the Bochner curvature tensor, respectively, so that $\| R \|^2 = 16 \Sigma R^i_{jk\bar{l}} R^j_{il\bar{k}}$, $\| S \|^2 = 2\Sigma R_{i\bar{j}} R_{j\bar{i}}$, and $\| B \|^2 = 16\Sigma B^i_{jk\bar{l}} B^j_{il\bar{k}}$. It is easily seen that

$$\| B \|^2 = \| R \|^2 - \frac{8}{n+2} \| S \|^2 + \frac{2}{(n+1)(n+2)} \rho^2. \tag{1}$$

We have the following

LEMMA [3]. *Let* M *be an n-dimensional Kaehler manifold. Then*

$$\frac{n(n+1)}{2} \| R \|^2 \geqq 2n \| S \|^2 \geqq \rho^2.$$

The first equality holds if and only if M *is a complex space form and the second equality holds if and only if* M *is Einsteinian.*

§ 3. PROOF OF THEOREM 1

Let M be a compact n-dimensional Bochner-Kaehler manifold. If we define a closed $2k$-form γ_k by

$$\gamma_k = \frac{(-1)^k}{(2\pi\sqrt{-1})^k\, k!} \Sigma \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} \Omega^{j_1}_{i_1} \wedge \cdots \wedge \Omega^{j_k}_{i_k},$$

then the k-th Chern class c_k of M is represented by γ_k. In particular, c_1 and c_2 are represented by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi} \Sigma \Omega_i^i ,$$

and

$$\gamma_2 = -\frac{1}{8\pi^2} \Sigma (\Omega_i^i \wedge \Omega_j^j - \Omega_i^j \wedge \Omega_j^i) ,$$

respectively. Thus, we see that $\omega^{n-2} c_1^2$ and $\omega^{n-2} c_2$ are represented by

$$\Phi^{n-2} \gamma_1^2 \quad \text{and} \quad \Phi^{n-2} \gamma_2 ,$$

respectively. Hence we have

$$\omega^{n-2} c_1^2 = \int_M (* \Phi^{n-2} \gamma_1^2) * 1$$

and

$$\omega^{n-2} c_2 = \int_M (* \Phi^{n-2} \gamma_2) * 1 ,$$

where $*$ denotes the Hodge star operator. Thus by straight-forward computation we get

$$\omega^{n-2} c_1^2 = \frac{(n-2)!}{16\pi^2} \int_M (\rho^2 - 2 \| S \|^2) * 1 , \tag{2}$$

$$\omega^{n-2} c_2 = \frac{(n-2)!}{32\pi^2} \int_M (\rho^2 - 4 \| S \| + \| R \|^2) * 1 . \tag{3}$$

By substituting (1) into (3) we get

$$\omega^{n-2} c_2 = \frac{(n-2)!}{32\pi^2} \int_M \left[\frac{n(n+3)}{(n+1)(n+2)} \rho^2 - \frac{4n}{n+2} \| S \|^2 \right] * 1 . \tag{4}$$

From (2) and (4) we have

$$\omega^{n-2} c_2 - \frac{n}{2(n+1)} \omega^{n-1} c_1^2 = \frac{(n-2)!\, n}{32(n+1)(n+2)\pi^2} \int_M (\rho^2 - 2n \| S \|^2) * 1 .$$

Thus, by using Lemma, we get

$$\omega^{n-2} c_2 \leqq \frac{n}{2(n+1)} \omega^{n-2} c_1^2 . \tag{5}$$

On the other hand, by using (2) and (4) again we have

$$\omega^{n-2} c_2 - \frac{n}{n+2} \omega^{n-2} c_1^2 = \frac{-n!}{32(n+1)(n+2)\pi^2} \int_M \rho^2 * 1 \leqq 0,$$

which implies

$$\omega^{n-2} c_2 \leqq \frac{n}{n+2} \omega^{n-2} c_1^2 . \tag{6}$$

Since $\frac{n}{2(n+1)} \leqq \frac{n}{n+2}$ for $n \geqq 2$, (5) and (6) show that $\omega^{n-2} c_2 \leqq$ $\leqq \frac{n}{2(n+1)} \omega^{n-2} c_2$ if $\omega^{n-2} c_1^2 \geqq 0$ and $\omega^{n-2} c_2 \leqq \frac{n}{n+2} \omega^{n-2} c_1^2$ if $\omega^{n-2} c_1^2 < 0$. The remaining part of the theorem follows immediately from Theorem 3 of [3]. (Q.E.D.)

§ 4. APPLICATIONS

A Kaehler manifold M is said to be *cohomologically Einsteinian* if $c_1 = a\omega$ for some constant a. Since $\omega^n = [\Phi^n]$ and $\Phi^n = n! * 1$, $\omega^{n-2} c_1^2 \geqq 0$ for any cohomologically Einstein-Kaehler manifold. Thus Theorem 1 implies immediately the following

COROLLARY 1. *Let* M *be a compact n-dimensional cohomologically Einstein Bochner-Kaehler manifold. If* $\omega^{n-2} c_2 \geqq \frac{n}{2(n+1)} \omega^{n-2} c_1^2$, *then* M *is a complex space form.*

Corollary 1 generalizes Theorem 1 of [4].

Combining Corollary 1 with Theorem 1 of [3] we have the following

COROLLARY 2 [5]. *Let* M *be a compact Einstein Bochner-Kaehler manifold. Then* M *is a complex space form.*

Let τ denote the Hirzebruch signature of a Kaehler surface M. Then we have $\tau = 1/3 (c_1^2 - 2 c_2)$. Thus from Theorem 1 (a) we have the following

COROLLARY 3 [2]. *Let* M *be a compact Bochner-Kaehler surface. Then* $\tau \geqq 0$.

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