Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Sahib Ram Mandan

On a Gerber's Conjecture

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.5, p. 411–419.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_5_411_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Geometria. — *On a Gerber's Conjecture.* Nota di SAHIB RAM MANDAN, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabiliti vari risultati inerenti ad una coppia di $(n+1)$–simplessi riferriti fra loro e situati in uno spazio euclideo o proiettivo ad n dimensioni.

In a letter [5] Gerber writes: " I conjecture the truth of the following statement which would be a fitting complement to the result [14] announced in youy letter of 25th March.

" Let p be a prime in n-dimensional Euclidean space E_n, (A) and (B) simplexes with x^i $(x = a, b)$ as faces opposite their vertices X_i $(X = A, B)$, and X'_i orthogonal projections of X_i on p. If the perpendiculars from A'_i to b^i concur (are associated), then those from B'_i to a^i behave the same way ".

It leads to a PORISM as follows:

If x^i_1 $(x = a, b; i = 0, \cdots, n)$ *are the* 2 *sets of normals to a prime* p *in* E_n *from* 2 *general sets* (X') *of points* X'_i $(X = A, B)$ *on* p *and* (X) *a pair of simplexes with vertices* X_i *on* x^i_1 *and faces* x^i *opposite* X_i *such that the* $n+1$ *normals to* b^i *from* A'_i *concur or form an associated set with* $(n-2)$*-parameter family of* $(n-2)$*-flats meeting them, then it is true for every member of the* $(n+1)$*-parameter family* f(B) *of simplexes like* (B), *and the* $n+1$ *normals from* B'_i *to the faces* a^i *of any member of the* $(n+1)$*-parameter family* f(A) *of simplexes like* (A) *behave the same way.* An associated set of lines are said to be in Schläfli position ([15], p. 248).

The purpose of this paper is then to prove the porism from which Gerber's Conjecture follows, and the existence in E_n $(2 < n)$ of (i) *Orthological Sets* (X') such that each join $A'_i A'_j$ is normal to the $(n-2)$-flat determined by B'_k $(k \neq i, j)$, and (ii) *Skew Orthological Sets* (X') such that the $n+1$ pairs of corresponding $(n-1)$-simplexes formed of them are skew orthological ([4]; [14]). The projective equivalent of the porism and its extension in n-dimensional projective spaces S_n for all values of n are also given besides an immediate deduction of a partly new result.

1. THE PLANE PORISM PICTURE

The porism in E_2 leads us to the following

THEOREM 1. *In* E_2 *if* x^i_1 $(x = a, b; i = 0, 1, 2)$ *are the* 2 *triads of perpendiculars to a line* p *from* 2 *triads of points* X'_i $(X = A, B)$ *on* p *and* (X) *a pair of triangles with vertices* X_i *on* x^i_1 *and sides* x^i *opposite* X_i *such*

(*) Nella seduta del 13 novembre 1976.

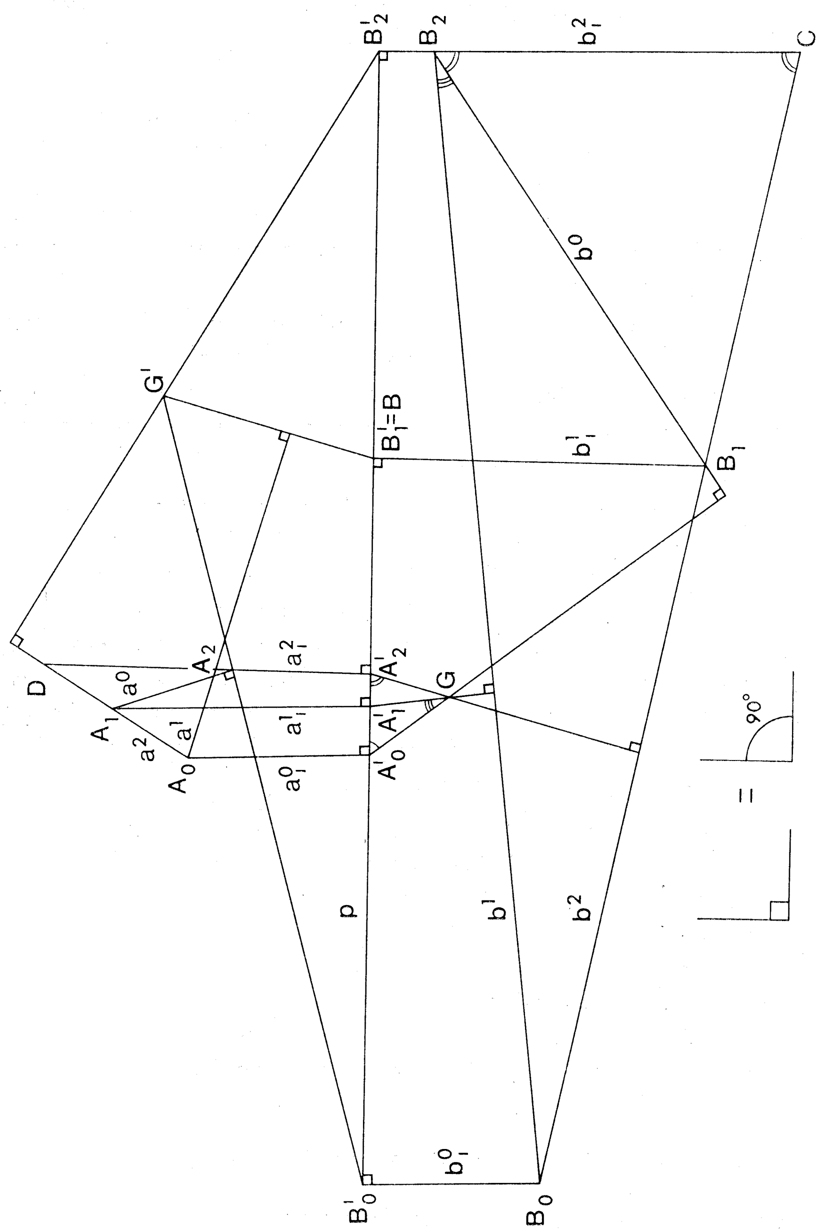

B'2
B2
b²1
C
b⁰
G'
B'1=B
b¹1
B1
D
a⁰
A2
a²1
A'2
G
A1
a¹1
A'1
a²
a¹
A0
a⁰1
A'0
p
b¹
b²
b⁰1
B'0
B0
90°
=

Fig. 1.

that the 3 perpendiculars to b^i from A'_i concur at a point G, then it is true for every member of the 3-parameter family f(B) of triangles like (B), and the 3 perpendiculars from B'_i to the sides a^i of any member of the 3-parameter family f(A) like (A) concur at a point G′ if and only if $A'_0 A'_1/A'_1 A'_2 = B'_0 B'_1/B'_1 B'_2$.

Proof. Fig. 1 shows that if the perpendiculars from A'_i to b^i concur at G, we have

$$\begin{aligned} A'_0 A'_1/A'_1 A'_2 &= \sin A'_0 G A'_1 \cdot \sin G A'_2 A'_1/(\sin A'_1 G A'_2 \cdot \sin G A'_0 A'_1) \\ &= \sin B_0 B_2 B_1 \cdot \sin C/(\sin B_2 B_0 B_1 \cdot \sin B_1 B_2 C) \\ &= B_0 B_1/B_1 C \quad (\text{C meet of } B_0 B_1 \text{ and } B_2 B'_2) \\ &= B'_0 B'_1/B'_1 B'_2, \end{aligned}$$

a result independent of (B), that is, it is true for all (B) with vertices B_i on b^i_1 independent of one another, every vertex having an infinity of choices.

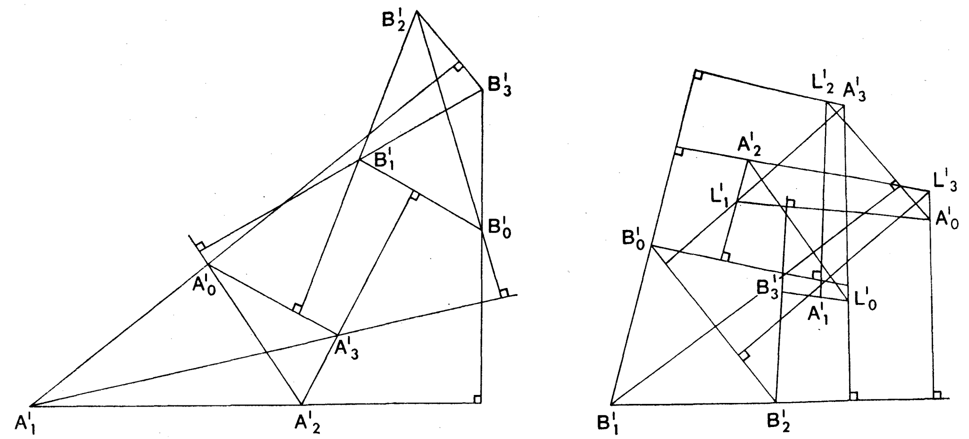
Now if the perpendiculars from B'_0, B'_2 to a^0, a^2 meet at G′ and one from G′ to a^1 meets p at B, by a similar agument we have $B'_0 B/BB'_2 = A'_0 A'_1/A'_1 A'_2$ that is then true if and only if $B = B'_1$.

2. ORTHOLOGICAL AND SKEW ORTHOLOGICAL SETS (X′)

The porism in E_3 leads us to the following

THEOREM 2. *In E_3 if x^i_1 $(x = a, b; i = 0, 1, 2, 3)$ are the 2 tetrads of normals to a plane p from the vertices X'_i $(X = A, B)$ of 2 quadrangles (X′) in p and (X) a pair of tetrahedra with vertices X_i on x^i_1 and faces x^i opposite X_i such that the 4 normals to b^i from A'_1* (i) *concur at a point* G, *or,* (ii) *lie in a regulus, then it is true for every member of the 4-parameter family f(B) of tetrahedra like* (B), *and the 4 normals from B'_i to the faces a^i of any member of the 4-parameter family f(A) of tetrahedra like* (A) (i) *concur at a point* G′, *or,* (ii) *lie in a regulus if and only if* (X′) *are* (i) *orthological* such that each side of one is perpendicular to the corresponding opposite side of the other as in fig. 2 (i), or, (ii) *skew orthological such that each pair of their corresponding triangles are orthological unlike* (i) as shown in fig. 2 (ii) where $L'_i \neq A'_i$ is the point of concurrence of the perpendiculars from A'_j, A'_k, A'_m to $B'_k B'_m$, $B'_m B'_j$, $B'_j B'_k$ $(i, j, k, m = 0, 1, 2, 3)$.

Proof. It follows from that of the following Theorem 3 by putting $n = 3$ there and noting that lines in a regulus are met by at least 3 lines of its complementary one, and there are no skew orthological triangles which may be orthological only.



(*i*) Orthologica Quadrangles (x') (*ii*) Skew Orthological Quadrangles (x')

Fig. 2.

The porism in E_n $(2 < n)$ leads us to the following

THEOREM 3. *The porism in* E_n $(2 < n)$ *is true if and only if the given* 2 *sets of points* (X') *on the given prime* p *are* (i) *orthological, or,* (ii) *skew orthological.*

Proof. Let A''_i be the foot of the normal from A'_i to b^i and A'_{ij} $(\neq A'_{ji})$ the meet of the normal from A''_i to b^j with p. Then the plane $A'_i A''_i A'_{ij}$ and similarly $A'_j A''_j A'_{ji}$ are perpendicular to the common $(n-2)$-flat b^{ij}_{n-2} of the pair of faces b^i, b^j of the simplex (B) and therefore perpendicular to any prime through this flat, in particular to the prime b^{ij} determined by the $n-1$ normals $b^k_1 = B_k B'_k$ $(k \neq i, j)$ from the vertices B_k of (B) to p and hence perpendicular to p. Consequently the 2 joins $A'_i A'_{ij}$, $A'_j A'_{ji}$ are both normal to b^{ij} and therefore to its $(n-2)$-flat $(b^{ij}_{n-2})'$ in p determined by the $n-1$ points B'_k there. Now there arise 2 cases.

(i) *If the* $n+1$ *normals from the points of the set* (A') *to the corresponding faces of* (B) *concur* at a point G, the plane $GA'_i A'_j$ determined by 2 normals $GA'_i A''_i$, $GA'_j A''_j$ contains their parallels $A''_j A'_{ji}$, $A''_i A'_{ij}$ and meets p in a line where then colline the tetrad of points: $A'_{ji}, A'_i, A'_j, A'_{ij}$. Or, the join of any 2 points A'_i, A'_j of (A') contains both A'_{ij}, A'_{ji} and is then normal to the corresponding $(n-2)$-flat $(b^{ij}_{n-2})'$ of (B') such that the $n(n+1)/2$ joins $B'_i B'_j$ of (B') are normal respectively to the corresponding $(n-2)$-flats $(a^{ij}_{n-2})'$ of (A'). Such a mutual relation between the 2 sets (X') makes them independent of (B). That is, the Theorem is true for every member of the family f(B) if it is so for one.

Again, when (X') are so related, $B'_i B'_j$ is normal to the prime a^{ij} determined by the $n-1$ normals $a^k_1 = A_k A'_k$ to p and therefore perpendicular to the $(n-2)$-flat a^{ij}_{n-2} of the simplex (A) common to its faces a^i, a^j

determined by its $n-1$ vertices A_k $(k \neq i, j)$ such that the normal from B_i' to a^i and $B_i' B_j'$ determine a plane perpendicular to this flat and that then contains the normal from B_j' to a^j, or the 2 normals meet. Thus all the $n+1$ normals from the points of (B') to the corresponding faces of any member (A) of the family $f(A)$ of simplexes meet one another and hence concur as desired.

(ii) *If the* $n+1$ *normals from the points of* (A') *to the corresponding faces of* (B) *form an associated set* (lie in a regulus in E_3 and concur in E_2), there exist a $(n-2)$-parameter family of $(n-2)$-flats meeting them and therefore a $(n-3)$-parameter family (unique line in E_3) of them parallel to each normal such that one parallel to $A_i' A_i''$ meets all other n normals $A_j' A_j''$, is parallel to the n joins $A_j'' A_{ji}'$ and therefore coprimal with the n planes $A_j' A_j'' A_{ji}'$, or, meets the n joins $A_j' A_{ji}'$ which then meet their $(n-3)$-parameter family of $(n-3)$-flats (unique point in E_3) in p and form an associated set by definition ([1], pp. 120–23 and [8] for $n = 4, 5$; [4]; [9]; [12]; [14]). That is, the n normals from the n vertices A_j' of the $(n-1)$-simplex $(a^i)'$ formed of the n points of (A') other than A_i' to the corresponding $(n-2)$-flats $(b_{n-2}^{ij})'$ of the $(n-1)$-simplex $(b^i)'$ formed of the n points B_j' of (B') other than B_i' form an associated set and therefore makes these $2\,(n-1)$-simplexes *skew orthological* ([4]; [15]). Such a relation of the 2 sets (X') is obviously independent of (B), or the Theorem is true for every member of the family $f(B)$ of simplexes if it is so for one.

Again, if B_i'' is the foot of the normal from B_i' to a^i and B_{ij}' $(\neq B_{ji}')$ the meet of the normal form B_i'' to a^j with p, we can prove that the 2 joins $B_i' B_{ij}'$, $B_j' B_{ji}'$ are both normal to the $(n-2)$-flat (a_{n-2}^{ij}) determined by the $n-1$ points A_k $(k \neq i, j)$ by interchanging the roles of A, a with B, b in the above argument. Consequently, by the mutual relation of (X'), the n normals $B_j' B_{ji}'$ from the n vertices of $(b^i)'$ to the corresponding $(n-2)$-flats $(a_{n-2}^{ij})'$ of $(a^i)'$ form an associated set and are met by $(n-3)$-parameter family of $(n-3)$-flats. Hence there exists a $(n-3)$-parameter family of $(n-2)$-flats, parallel to $B_i' B_i''$ and therefore to the n joins $B_j'' B_{ji}'$ or comprimal with the n planes $B_j' B_j'' B_{ji}'$, which then meet the n normals $B_j' B_j''$ from B_j' to a^j. Or, the $n+1$ normals from the $n+1$ points of (B') to the corresponding faces of any member of the family $f(A)$ of simplexes form an associated set, as desired, by a Lemma, established in 1965 [11] and used later in [14], that runs as follows:

If through the $n+1$ *vertices of a simplex* S *in* S_n $n+1$ *lines are drawn such that there pass a* $(n-3)$*-parameter family of* $(n-2)$*-flats through each vertex to meet them, the lines then form an associated set.*

Its proof given there holds good also in E_n even for a degenerate S whose vertices may lie in a prime which is one at infinity in the present case.

3. Projective Equivalent of Porism

A line x_1^i is said to be normal or perpendicular to a prime p in S_n if its meet P with a fixed prime a (said to be at infinity in E_n) is pole of p (or of common secondum of p, a) for a fixed quadric W (called an *Absolute or a sphere at infinity*) in a. Thus the projective equivalent of the porism and Theorems 1-3 takes the shape of the following

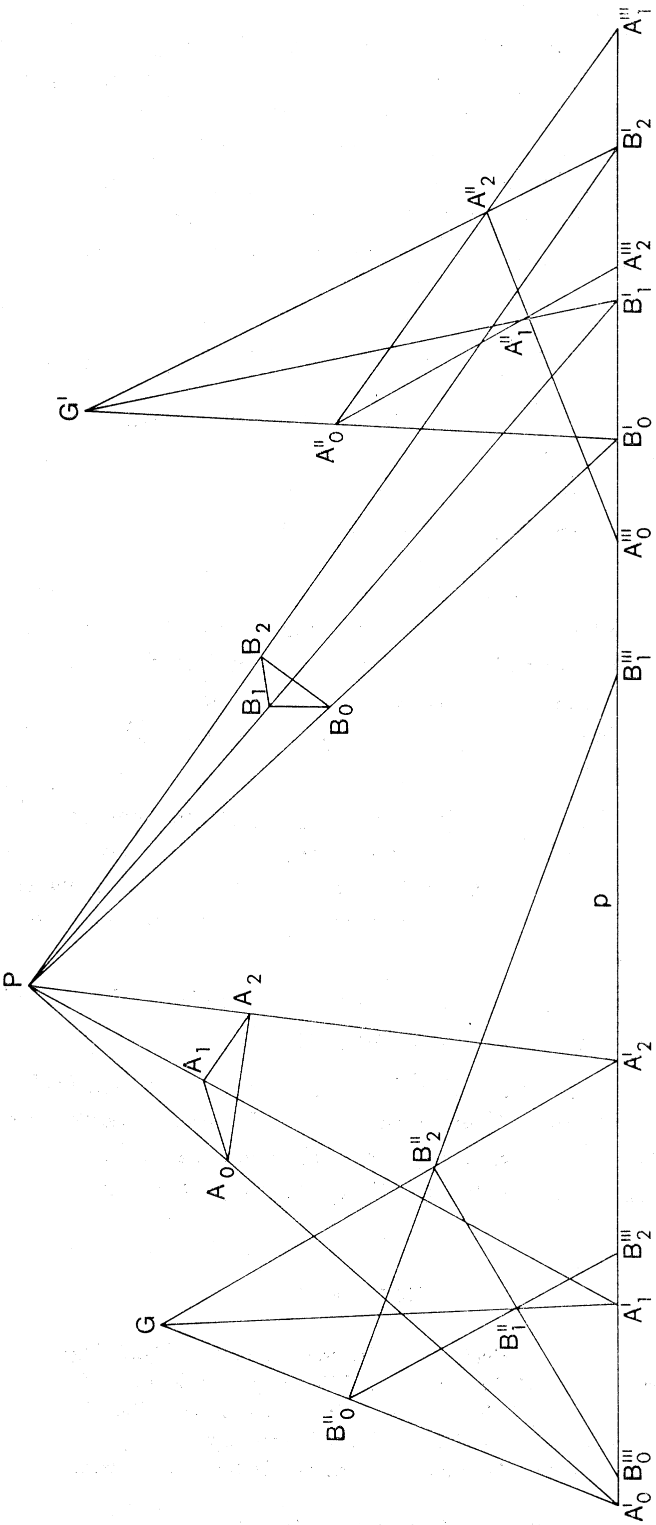
Porism P. *In* S_n *if* x_1^i $(x = a, b; i = 0, \cdots, n)$ *are* 2 *sets of joins of* 2 *general sets* (X′) *of points* X_i' $(X = A, B)$ *on a prime* p *to its pole* P *for a quadric* W (*a pair of points* W″, W‴ *in* S_2 *and a conic* W *in* S_3) *in* a *fixed prime* a, (X) *a pair of simplexes* (*triangles in* S_2 *and tetrahedra in* S_3) *with vertices* X_i *on* x_1^i *and faces* (*sides in* S_2) x^i *opposite* X_i *and* X_i'' *are poles of* x^i *in* a *for* W *such that the* $n+1$ *joins* $A_i' B_i''$ (i) *concur at a point* G, *or*, (ii) *form an associated set* (*if* $2 < n$), *it is true for every member of the* $(n+1)$-*parameter family* f(B) *of simplexes like* (B), *and the* $n+1$ *joins* $B_i' A_i''$ *behave the same way for every member of the* $(n+1)$-*parameter family* f(A) *of simplexes like* (A) *if and only if in* S_2 *the* 2 *cross ratios* $(X_0' X_1', X_2' A')$ *on the line* p *are equal with* A′ *as the common point of the* 2 *lines* $a = W'' W'''$ *and* p, *and in* S_n $(2 < n)$ (X′) *are 'projectively'* (i) *orthological such the each join* $A_i' A_j'$ *is conjugate to* $(n-2)$-*flat* $(b_{n-2}^{ij})'$ *determined by the* $n-1$ *points* B_k' $(k \neq i, j)$ *for* W, *or*, (ii) *skew orthological such that the* $n+1$ *pairs of corresponding* $(n-1)$-*simplexes formed of* (X′) *are projectively skew orthological* [14] in the sense that the $n+1$ joins of vertices of one simplex in a pair to the poles for W of the corresponding faces of the other form an associated set.

Proof. It is left *as an exercise*.

4. An Extension of Porism

It is interesting to note that the Porism P is true even if the quadric W in a prime a is replaced by a hyperquadric in S_n (W″, W‴ by a conic and conic W by a quadric) with certain noteworthy modifications in S_2 only as enunciated in the following

Theorem 1 P. *In* S_2 *if* P *is the pole of a line* p *for a conic* W, (X′) 2 *triads of points* X_i' $(X = A, B; i = 0, 1, 2)$ *on* p, (X) *a pair of triangles with vertices* X_i *on the joins* $x_1^i = PX_i'$ *and* (X″) *their polar triangles for* W *such that the* 3 *joins* $A_i' B_i''$ *concur at a point* G, *then it is true for any member of the* 3-*parameter family* f(B) *of triangles like* (B), *and the* 3 *joins* $B_i' A_i''$ *concur at a point* G′ *for any member of the* 3-*parameter family* f(A) *of triangles like* (A) *if and only if there exist the quadrangular set* $Q(A_0' A_1' A_2', B_0''' B_1''' B_2''')$ *leading to* $Q(B_0' B_1' B_2', A_0''' A_1''' A_2''')$ *where* X_i''' *are poles of* PX_i' *for* W.



P
A0
A1
A2
B0
B1
B2
G
G'
p
A'0
A'1
A'2
B'0
B'1
B'2
A''0
A''1
A''2
B''0
B''1
B''2
A'''0
A'''2
B'''0
B'''1
B'''2

Fig. 1 P.

Proof. X_i''' are obviously on p and on the sides $X_j''X_k''$ $(j, k = 0, 1, 2)$ of the triangles (X'') as shown in fig. 1 P giving rise to $Q(A_0'A_1'A_2', B_0'''B_1'''B_2''')$ on p by the quadrangle $GB_0''B_1''B_2''$ ([3], p. 240). Again this quadrangular set is projective to the set of conjugates on p of the 6 points there for W ([16], p. 119) leading to another such set $Q(A_0'''A_1'''A_2''', B_0'B_1'B_2')$ on p and that is equivalent to $Q(B_0'B_1'B_2', A_0'''A_1'''A_2''')$ in any Pappian plane ([3], p. 241). Consequently $B_i'A_i''$ must concur at a point G' to form a quadrangle $G'A_0''A_1''A_2''$ to give us the last quadrangular set.

5. An Immediate Deduction

Any 2 simplexes are said to be orthological or skew orthological according as the normals from the vertices of one to the corresponding faces of the other in a correspondence concur or form an associated set in E_{n-1} $(2 < n)$ such that an *orthocentric or orthogonal simplex* (whose altitudes concur at its orthocentre H) *is always orthological to itself*, or, an *orthocentric group* ([2], p. 320; [7]; [10]) or set formed of H and its vertices *is always orthological to itself*, and any *set of* $n+1$ *general points is always skew orthological to itself*. For the altitudes of a general simplex form an associated set ([4]; [8]; [12]). Thus the condition of the porism in Theorems 2–3 is automatically satisfied if $(A') = (B')$ and therefore $f(A) = f(B)$ leading to the following.

THEOREM G (cf. [14]). *If* x_1^i $(i = 0, \cdots, n)$ *are normals to a prime* p *from the points* X_i' *of a set* (X') *on* p *in* E_n $(2 < n)$ *and* (X) *any simplex with vertices* X_i *on* x_1^i *and faces* x^i *opposite* X_i, *the* $n+1$ *normals from* X_i' *to* x^i *form an associated set that reduces to a concurrent one if and only if* (X') *is orthocentric.*

On identification of (A') with (B') and therefore of $f(A)$ with $f(B)$ in Theorem 1 we have the following well known.

THEOREM O (cf. [6]). *If* x_1^i $(i = 0, 1, 2)$ *are perpendiculars to a line* p *from a triad of points* X_i' *on* p *in* E_2 *and* (X) *any triangle with vertices* X_i *on* x_1^i *and sides* x^i *opposite* X_i, *the* 3 *perpendiculars from* X_i' *to* x^i *concur at a point* o, called the *orthopole* ([2a], p. 287) of p for (X).

References

[1] H. F. Baker (1940) – *Principles of Geometry*, Vol. IV. (Cambridge).

[2] N. A. Court (1964) – *Modern Pure Solid Geometry*. (Chelsea).

[2 a] N. A. Court (1952) – *College Geometry*. (Barnes and Noble, N.Y.).

[3] H. S. M. Coxeter (1969) – *Introduction to Geometry*. (Wiley).

[4] L. Gerber (1975) – *Associated and Perspective Simplexes*, « Transac. Amer. Math. Soc. », *201*, 43–55.

[5] L. Gerber (1975) – *Personal letter dated April 22*.

[6] S. R. MANDAN (1946) – *Extension of Orthopole and Ortholine*, « J. Lahore Phil. Soc. », *8*, 60–61.
[7] S. R. MANDAN (1958) – *Semi-orthocentric and Orthogonal Simplexes in 4-Space*, « Supplement Bul. Cal. Math. Soc. », *50*, 21–29.
[8] S. R. MANDAN (1958) – *Altitudes of A General Simplex in 4-Space*, « Supplement Bul. Cal. Math. Soc. », *50*, 30–35.
[9] S. R. MANDAN (1959) – *Polarity for A Quadric in An n-space*, « Rev. Fac. Scs. Uni, Istanbul », *A 24*, 21–40.
[10] S. R. MANDAN (1962) – *Altitudes of A Simplex in n-space*, « J. Austral. Math. Soc. », *5*, 403–24.
[11] S. R. MANDAN (1965) – *Pascal's Theorem in n-space*, « J. Austral. Math. Soc. », *5*, 401–8.
[12] S. R. MANDAN (1965) – *Altitudes of A General n-simplex*, « J. Austral. Math. Soc. », *5*, 409–15.
[13] S. R. MANDAN (1965) – *Concurrent Lines Related to 2 Plane Triangles*, « J. Sc. and Engg. Res. », *9*, 47–50.
[14] S. R. MANDAN – *Skew Orthological Perspective Simplexes* (to appear).
[15] B. SEGRE (1961) – *Lectures on Modern Geometry*. (Roma).
[16] A. SEIDENBERG (1962) – *Projective Geometry*. (Nostrand).