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Two density properties of ordinary differential equations

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Equazioni differenziali ordinarie. — *Two density properties of ordinary differential equations* (*). Nota di FRANCESCO S. DE BLASI (**), e JÓZEF MYJAK (***), presentata (****) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che arbitrariamente vicino ad ogni equazione differenziale in c_0 (§ 1) ne esiste almeno una per cui il corrispondente problema di Cauchy (1) è sprovvisto di soluzioni. Similmente, arbitrariamente vicino ad ogni equazione differenziale in l_p (§ 2) ne esiste almeno una per cui le successive approssimazioni (3), relative al problema di Cauchy (2), non convergono.

1. DIFFERENTIAL EQUATIONS IN c_0 WITHOUT EXISTENCE ARE DENSE

Denote by:

c_0 the Banach space of all sequences $x = (x_1, x_2, \cdots), x_n$ real, such that $\lim_{n\to\infty} x_n = 0$ with the supremum norm $\|x\| = \sup\{|x_n| : n \geq 1\}$;

E a nonempty open subset of c_0;

$C(E, c_0)$ the Banach space of all continuous and bounded functions $f: E \to c_0$ with the norm $\|f\|_E = \sup\{\|f(x)\| : x \in E\}$.

Let $x^0 \in c_0$ be given. For any $f \in C(E, c_0)$ consider the Cauchy problem

$$x' = f(x) \quad , \quad x(0) = x^0 \quad (' = d/dt). \tag{1}$$

In [3] Dieudonné constructed an example of an $f \in C(E, c_0)$ for which the Cauchy problem (1) has no solution. On the other hand it is a consequence of a theorem of Lasota and Yorke [6] (see also [2]) that the set N consisting of all $f \in C(E, c_0)$ for which problem (1) has no solution is of first category. Thus, since N is nonempty and meager, the natural question arises: how is N scattered in $C(E, c_0)$? In this section it will be proved that N is dense in $C(E, c_0)$.

A similar question could be raised if c_0 is replaced by any other concrete infinite dimensional Banach space, for instance by any l_p space or, more generally, by any abstract infinite dimensional Banach space X. More carefully, let U be a nonempty open subset of $\mathbf{R} \times X$. Let $C(U, X)$ be the

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Banach space of all continuous and bounded functions $f: U \to X$ with the supremum norm. Let $\mathcal{N}$ be the set of all $f \in C(U, X)$ for which the Cauchy problem $x' = f(t, x)$, $x(t_0) = x^0$, $(t_0, x^0) \in U$ has no solution. Combining recent results of Godunov [5] (see also [1]) and Lasota and Yorke [6] it turns out that $\mathcal{N}$ is nonempty and meager in $C(U, X)$. The authors tried (without success) to answer the question: is $\mathcal{N}$ dense in $C(U, X)$?

THEOREM 1. *The set* N *is dense in* $C(E, c_0)$.

Proof. If $x^0 \in E$ and $f \in C(E, c_0)$ are given, the Cauchy problem (1) will be denoted, for brevity, by $[f; x^0]$. Let $x^0 \in E$ be fixed. Let $f \in C(E, c_0)$ be such that the problem $[f; x^0]$ has a solution. Let $\varepsilon > 0$. It is claimed that there exists $g \in C(E, c_0)$ such that $\|g - f\|_E \leq \varepsilon$ and $[g; x^0]$ has no solution.

Set $y^0 = f(x^0)$, $y^0 = (y_1^0, y_2^0, \cdots)$ and consider the two cases:

Case 1. The sequence $\{y_n^0\}$ contains a subsequence $\{y_{n_k}^0\}$ of numbers $y_{n_k}^0 \geq 0$. Let $\eta = (\eta_1, \eta_2, \cdots) \in c_0$ be such that $\eta_n > 0$ and $\|\eta\| \leq \varepsilon/4$. Define the continuous function $\gamma_1 : E \to c_0$ by

$$\gamma_1(x) = \left(\sqrt{|x_1 - x_1^0|} + y_1^0 + \eta_1, \sqrt{|x_2 - x_2^0|} + y_2^0 + \eta_2, \cdots\right).$$

Let us show that the problem $[\gamma_1; x^0]$ has no solution. Suppose the contrary and let $u: [0, T] \to E$ $(T > 0)$ be a solution of $[\gamma_1; x^0]$. Then, for every $k = 1, 2, \cdots$, we have

$$u'_{n_k}(t) = \sqrt{|u_{n_k}(t) - x_{n_k}^0|} + y_{n_k}^0 + \eta_{n_k} \quad , \quad u_{n_k}(0) = x_{n_k}^0$$

and, since $y_{n_k}^0 + \eta_{n_k} > 0$, the function u_{n_k} is greater than or equal to the right maximal solution of the Cauchy problem

$$v' = \sqrt{|v - x_{n_k}^0|} \quad , \quad v(0) = x_{n_k}^0$$

that is, $u_{n_k}(t) \geq x_{n_k}^0 + \frac{1}{4} t^2$, $t \in [0, T]$. This implies $u(t) \notin c_0$ for $t \in (0, T]$, a contradiction.

Case 2. If the first case does not hold there exists a natural number n_0 such that $y_n^0 < 0$ for every $n \geq n_0$. Define the continuous function $\gamma_2 : E \to c_0$ by

$$\gamma_2(x) = \left(-\sqrt{|x_1 - x_1^0|} + y_1^0 \quad , \quad -\sqrt{|x_2 - x_2^0|} + y_2^0, \cdots\right)$$

and let us show that $[\gamma_2; x^0]$ has no solution. Otherwise, if u is a solution of this problem, for every $n \geq n_0$ we have

$$u'_n(t) = -\sqrt{|u_n(t) - x_n^0|} + y_n^0 \quad , \quad u_n(0) = x_n^0$$

and, since $y_n^0 < 0$, the function u_n is less than or equal to the right minimal solution of the Cauchy problem

$$v' = -\sqrt{|v - x_n^0|} \quad , \quad v(0) = x_n^0$$

that is, $u_n(t) \leq x_n^0 - \frac{1}{4} t^2$, $t \in [0, T]$, which yields again a contradiction.

Now, let $\gamma : E \to c_0$ be equal to γ_1, if Case 1 holds, otherwise let $\gamma = \gamma_2$. Since f and γ are continuous and for $x = x^0$ they satisfy $\|f(x^0) - \gamma(x^0)\| \leq \varepsilon/4$, there exists $\delta > 0$ such that for each x in the closed ball $B(x^0, \delta) \subset E$ we have

$$\|f(x) - \gamma(x)\| \leq \varepsilon.$$

Define $\varphi : B(x^0, \delta) \to c_0$ by $\varphi(x) = f(x) - \gamma(x)$. By Dugundji's theorem [4] there exists a continuous extension $\tilde{\varphi}$ of φ, defined on E and with range in c_0, such that $\|\tilde{\varphi}(x)\| \leq \varepsilon$ for each $x \in E$. Define $g : E \to c_0$ by

$$g(x) = f(x) - \tilde{\varphi}(x) \qquad x \in E,$$

and observe that $g \in C(E, c_0)$ and $\|g - f\|_E \leq \varepsilon$. Since $[g; x^0]$ has no solution, the proof is complete.

2. Differential equations for which the successive approximations do not converge are dense

Let $f : I \times \mathbf{R}^m \to \mathbf{R}^m$ be continuous, where $I = [0, 1]$. Consider the Cauchy problem

$$x' = f(t, x) \quad , \quad x(0) = x^0. \tag{2}$$

It is well known that there exist continuous f's for which the sequence $\{\psi^n\}$ of the successive approximations

$$\psi^n(t) = x^0 + \int_0^t f(s, \psi^{n-1}(s))\, ds \qquad (n \geq 1), \ \psi^0(t) = \varphi(t) \tag{3}$$

does not converge for some starting continuous function φ (see [7]), and this can happen even when problem (2) has a unique solution. In a recent paper Vidossich [8] has shown that for most differential equations (in the sense of the Baire category) the successive aproximations do converge. The purpose of this section is to complete Vidossich's result by showing that the f's for which the successive approximations do not converge for every φ, though few, are neverthless dense.

Denote by:

B_r the closed ball $B(x^0, r) \subset \mathbf{R}^m$ $(r > 0)$;

$C(I, B_r)$ the set of all continuous functions from I to B_r;

M the space of all continuous functions $f : I \times B_r \to \mathbf{R}^m$, with norm $\|f\| = \sup\{|f(t, x)| : (t, x) \in I \times B_r\}$, such that $\|f\| \leq K$ $(0 < K \leq r)$;

N_0 the subset of M consisting of all $f \in M$ for which the successive approximations (3) do not converge for some starting function $\varphi \in C(I, B_r)$.

It was shown in [8] that N_0 is a first category set in M. Here we shall prove the following

THEOREM 2. *The set* N_0 *is dense in* M.

Proof. Let $g \in M$ and $\varepsilon > 0$. We claim that there exists $f \in M$ such that $\|f - g\| \leq \varepsilon$ and the corresponding successive approximations (3) do not converge for some convenient starting function $\varphi \in C(I, B_r)$. We suppose, without loss of generality, $\|g\| < K$ and $\varepsilon < K - \|g\|$.

Set $y^0 = g(0, x^0)$, $x^0 = (x_1^0, \cdots, x_m^0)$, $y^0 = (y_1^0, \cdots, y_m^0)$. Define $h_1 : \mathbf{R}^m \to \mathbf{R}$ by

$$h_1(t, x) = \begin{cases} y_1^0, & t = 0, \\ y_1^0 + 2t, & 0 < t \leq 1, \quad x_1 \leq y_1^0 t + x_1^0 \\ y_1^0 + 2t - \dfrac{4(x_1 - y_1^0 t - x_1^0)}{t}, & 0 < t \leq 1, \quad y_1^0 t + x_1^0 < x_1 < y_1^0 t + t^2 + x_1^0 \\ y_1^0 - 2t, & 0 < t \leq 1, \quad x_1 \geq y_1^0 t + t^2 + x_1^0. \end{cases}$$

It is easy to see that the function $h : I \times \mathbf{R}^m \to \mathbf{R}^m$ defined by $h(t, x) = (h_1(t, x), y_2^0, \cdots, y_m^0)$ is continuous. Since h and g agree at $(0, x^0)$ and are continuous, there exists a sufficiently small δ, $0 < \delta < \min\{1, r\}$ such that

$$|g(t, x) - h(t, x)| \leq \varepsilon \qquad \text{for} \quad (t, x) \in [0, \delta] \times B_\delta,$$

where $B_\delta = B(x^0, \delta)$. Define $\chi : [0, \delta] \times B_\delta \to \mathbf{R}^m$ by $\chi(t, x) = g(t, x) - h(t, x)$. By virtue of Dugundji's theorem there exists a continuous extension $\tilde{\chi}$ of χ, defined on $I \times B_r$, with values in $\mathbf{R}^m$, and satisfying $\|\tilde{\chi}\| \leq \varepsilon$. Define $f : I \times B_r \to \mathbf{R}^m$ by

$$f(t, x) = g(t, x) - \tilde{\chi}(t, x),$$

and observe that $f \in M$ and $\|f - g\| \leq \varepsilon$.

Let $\tilde{\varphi}(t) = (x_1^0 + y_1^0 t - t^2, x_2^0 + y_2^0 t, \cdots, x_m^0 + y_m^0 t)$ for $t \in [0, \delta^*]$, where $0 < \delta^* \leq \min\{\delta, \delta/K\}$ is such that $\tilde{\varphi}(t) \in B_\delta$, $t \in [0, \delta^*]$. Define $\varphi : I \to B_r$ putting $\varphi(t) = \tilde{\varphi}(t)$, $t \in [0, \delta^*]$ and $\varphi(t) = \tilde{\varphi}(\delta^*)$ elsewhere. Observe that $\varphi \in C(I, B_r)$.

Let $\{\psi^n\}$ be the sequence of successive approximations for f, with starting function φ, and let $\tilde{\psi}^n$ be the restriction of ψ^n to $[0, \delta^*]$. By an easy calculation one obtains for $k = 1, 2, \cdots$

$$\begin{aligned} \tilde{\psi}^{2k-1}(t) &= (x_1^0 + y_1^0 t + t^2, x_2^0 + y_2^0 t, \cdots, x_m^0 + y_m^0 t) \\ \tilde{\psi}^{2k}(t) &= (x_1^0 + y_1^0 t - t^2, x_2^0 + y_2^0 t, \cdots, x_m^0 + y_m^0 t). \end{aligned} \qquad t \in [0, \delta^*]$$

Since the sequence $(\tilde{\psi}^n\}$ does not converge, the same happens for $\{\psi^n\}$. Thus $f \in N_0$. This completes the proof.

The preceding theorem can be extended to any l_p space $1 \leq p \leq +\infty$.

THEOREM 3. *In the definition of* M, N_0 *let* $\mathbf{R}^m$ *be replaced by* l_p, $1 \leq p \leq +\infty$. *Then* N_0 *is dense in* M.

Proof. Let $g \in M$, $x^0 \in l_p$. Let $h(t, x) = (h_1(t, x), y_2^0, y_3^0, \cdots)$, where $y^0 = g(0, x^0)$ and $h_1(t, x)$ are defined as above, and use the argument of the proof of Theorem 2.

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