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An asymptotic analysis of nonlinear n-th order retarded differential equations

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Equazioni differenziali ordinarie. — *An asymptotic analysis of nonlinear n-th order retarded differential equations* (*). Nota di LU-SAN CHEN, CHEH-CHIH YEH e JER-SAN LIN, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si studia il comportamento asintotico di un'equazione differenziale ordinaria ad argomenti ritardati.

1. INTRODUCTION

The purpose of this paper is to establish asymptotic behavior of solutions for the retarded differential equations of the form

$$\text{(1)} \qquad L_n x(t) + f(t, x[g_1(t)], \cdots, x[g_m(t)]) = 0$$

where L_n is an operator defined recursively by

$$L_0 x(t) = x(t), L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t), \quad i = 1, 2, \cdots, n, \qquad r_n(t) = 1.$$

Results concerning the oscillatory behavior of a wide variety of retarded equations can be found in [1–7]. Throughout this paper, the following conditions always hold:

(i) $r_i(t) \in C[R_+ \equiv [0, \infty), R_+ \setminus \{0\}], \int^{\infty} r_i(t)\, dt = \infty, \; i = 1, 2, \cdots, n-1.$

(ii) $f(t, y_1, \cdots, y_m) \in C[R_+ \times R^m, R = (-\infty, \infty)], f(t, 0, \cdots, 0) = 0,$

(iii) $g_j(t) \in C[R_+, R], \lim_{t \to \infty} g_j(t) = \infty, \qquad j = 1, 2, \cdots, m.$

The monotonicity of f is considered with respect to the order in R^m defined as follows:

$$X = (x_1, \cdots, x_m) \leq Y = (y_1, \cdots, y_m) \Longleftrightarrow x_j \leq y_j$$

for $j = 1, 2, \cdots, m$. Let condition (ii) hold; we say that $f(t, x_1, \cdots, x_m)$ belongs to the class C^* if it is nondecreasing with respect to $x_1, \cdots, x_m$. And $f(t, x_1, \cdots, x_m)$ belongs to C^{**} if it is nonincreasing with respect to $x_1, \cdots, x_m$. For convenience, we define $\omega_i, i = 0, 1, \cdots, n-1$ as follows:

$$\omega_0(t) = 1, \quad \omega_1(t) = \int_0^t r_1(s)\, ds, \quad \omega_i(t) = \int_0^t r_i(s)\, \omega_{i-1}(s)\, ds.$$

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2. PRELIMINARY LEMMAS

In order to obtain our results, the following two Kiguradze's Lemmas [5, 6] are very important in our later treatment. Lemma 1 was extended by the authors [1]. We only prove Lemma 2.

LEMMA 1. *Let $u(t)$ be a positive n-times continuously differentiable function on an interval $[a, \infty)$. If $L_n u(t)$ is of constant sign and not identically zero for all large t, then there exist a $t_u \geq a$ and an integer k, $0 \leq k \leq n$ with $n + k$ odd if $L_n u(t) \leq 0$, $n + k$ even if $L_n u(t) \geq 0$ and such that for every $t \geq t_u$*

$$\begin{cases} L_i u(t) \geq 0, & i = 0, 1, \cdots, k-1 \\ (-1)^{n+i} L_i u(t) \geq 0, & i = k, k+1, \cdots, n. \end{cases} \tag{2}$$

LEMMA 2 (Kiguradze). *Let $x(t)$ be a solution of the differential equation*

$$L_n x(t) + f_1(t, x[g_1(t)], \cdots, x[g_m(t)]) = Q_1(t) \tag{3}$$

defined on the interval $(0, t_0)$, and let $y(t)$ be a nonnegative solution of the differential equation

$$L_n y(t) + f_2(t, y[g_1(t)], \cdots, y[g_m(t)]) = Q_2(t) \tag{4}$$

defined on this interval. If

$$\begin{cases} (-1)^{n-1}\{f_1(t, x[g_1(t)], \cdots, x[g_m(t)]) - f_2(t, x[g_1(t)], \cdots, x[g_m(t)])\} \geq 0, \\ \textit{for} \quad 0 \leq t \leq t_0, \quad x[g_j(t)] \geq y[g_j(t)], \qquad j = 1, 2, \cdots, m, \\ (-1)^k \{L_k x(t_0) - L_k y(t_0)\} > 0, \qquad k = 0, 1, \cdots, n-1, \\ (-1)^n \{Q_1(t) - Q_2(t)\} \geq 0 \quad \textit{for} \quad 0 \leq t \leq t_0. \end{cases} \tag{5}$$

then for $0 \leq t \leq t_0$

$$(-1)^k [L_k x(t) - L_k y(t)] > 0, \qquad k = 0, 1, \cdots, n. \tag{6}$$

Proof. From (3) and (4)

$$\begin{aligned} (-1)^k [L_k x(t) - L_k y(t)] &= \sum_{i=k}^{n-1} (-1)^i [L_i x(t_0) - L_i y(t_0)] \, \omega_{i-k}(t) \\ &+ (-1)^{n-1} \int_t^{t_0} \omega_{n-k-1}(s) \{f_1(s, x[g_1(s)], \cdots, x[g_m(s)]) - \\ &- f_2(s, y[g_1(s)], \cdots, y[g_m(s)])\} \, ds \\ &+ (-1)^n \int_t^{t_0} \omega_{n-k-1}(s) [Q_1(s) - Q_2(s)] \, ds. \end{aligned} \tag{7}$$

It follows from (5) that (6) are satisfied on some interval $(t^*, t_0]$; we shall show that $t^* = 0$. If $t^* \neq 0$, then we would have $L_{n-1} x(t^*) = L_{n-1} y(t^*)$. But this is impossible, since by (5) and (7) we have

$$(-1)^{n-1} [L_{n-1} x(t^*) - L_{n-1} y(t^*)] > (-1)^{n-1} [L_{n-1} x(t_0) - L_{n-1} y(t_0)] > 0 .$$

COROLLARY 1. *Let $x(t)$ be a solution of* (1), $c \geq 0$,

$$(-1)^k L_k x(c) > 0 \qquad \textit{for} \quad k = 0, 1, \cdots, n-1$$

and $(-1)^{n-1} f(t, x[g_1(t)], \cdots, x[g_m(t)]) \geq 0$ *for* $0 \leq t \leq c$.

Then $(-1)^k L_k x(t) > 0$ *for* $0 \leq t \leq c$ *and* $k = 0, 1, \cdots, n-1$.

3. MAIN RESULTS

A solution $x(t)$ of (1) is called strongly decreasing if there is $c \geq 0$ such that for $t \geq c$

$$(-1)^k L_k x(t) > 0, \qquad k = 0, 1, \cdots, n-1. \tag{8}$$

From Corollary 1 we can prove the following.

THEOREM 1. *Suppose that* $(-1)^{n-1} f(t, X) X \geq 0$, *where* $X = (x[g_1(t)], \cdots, x[g_m(t)])$ *for* $t \geq T$. *Let every eventually positive solution $x(t)$ of (1) be strongly decreasing. If $x(t)$ is a solution of (1) and any of $L_i x(t)$, $i = 1, 0, \cdots, n-1$, has a zero in $[0, \infty)$, then $x(t)$ is oscillatory.*

Proof. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1). Now there exists a $c \geq 0$ such that (8) holds for $t \geq c$ and $k = 0, 1, \cdots, n-1$. If follows from Corollary 1 that $L_k x(t)$ $k = 0, 1, \cdots, n-1$, has no zeros in $[0, c)$ and thus has no zeros at all.

THEOREM 2. *Let n be odd (even) and* $F \in C[R_+ \times R^m, R]$ *with*

$$F(t, x_1, \cdots, x_m) \geq f(t, x_1, \cdots, x_m)$$

for $t \geq 0$, $x_j \in R$, $j = 1, 2, \cdots, m$. *Suppose that* $F, f \in C^* (C^{**})$ *and every eventually positive solution of (1) is strongly decreasing. Then every eventually positive solution of*

$$L_n x(t) + F(t, x[g_1(t)], \cdots, x[g_m(t)]) = 0 \tag{9}$$

is strongly decreasing.

Proof. We only prove the case where n is odd. Let $y(t)$ be an eventually positive solution of (9) which is not strongly decreasing. Let $T \geq 0$ be such that $y[g_j(t)] > 0$ for $t \geq T$. Since $F \in C^*$, $L_n y(t) < 0$. From Lemma 1 we see that there is $c \geq T$ and an even integer $\tilde{j} \neq 0$ such that for $t \geq c$

$$L_i y(t) > 0, \qquad i = 0, 1, \cdots, \tilde{j}$$

and

$$(-1)^{n+i-1} L_i y(t) \geq 0, \qquad i = \tilde{j} + 1, \cdots, n.$$

Integrating (9) $n - \tilde{j}$ times we have

$$\text{(10)} \qquad L_{\tilde{j}} y(t) \geq \int_t^\infty r_{\tilde{j}+1}(s_{n-\tilde{j}-1}) \int_{s_{n-\tilde{j}-1}}^\infty r_{\tilde{j}+2}(s_{n-\tilde{j}-2}) \cdots$$

$$\cdots \int_{s_1}^\infty F(s, y[g_1(s)], \cdots, y[g_m(s)]) \, ds \, ds_1 \cdots ds_{n-\tilde{j}-1}$$

$$\equiv \varphi(t, F, y) \geq \varphi(t, f, y).$$

Integrating (10) $\tilde{j}$ times from c to $t \geq c$ we obtain

$$\text{(11)} \qquad y(t) \geq y(c) + \int_c^t r_1(u_{\tilde{j}-1}) \int_c^{u_{\tilde{j}-1}} r_2(u_{\tilde{j}-2}) \cdots$$

$$\cdots \int_c^{u_1} r_{\tilde{j}}(u) \, \varphi(u, f, y) \, du \, du_1 \cdots du_{\tilde{j}-1}$$

$$\equiv y(c) + \psi(t, f, y).$$

Now define

$$x_0(t) = y(t) \quad , \quad x_{\eta+1}(t) = y(c) + \psi(t, f, x_\eta), \quad \eta = 0, 1, \cdots$$

for $t \geq c$. From (11) we obtain by induction that for $t \geq c$ and $\eta = 0, 1, \cdots$

$$0 < x_\eta(t) \leq y(t),$$

$$x_{\eta+1}(t) \leq x_\eta(t).$$

Consequently, letting $\lim_{\eta \to \infty} x_\eta(t) = x(t)$, and applying Lebesgue's theorem on monotone convergence we get for $t \geq c$

$$\text{(12)} \qquad x(t) = x_0 + \psi(t, f, x).$$

Hence $x(t) \geq x_0$ for $t \geq c$, i.e. $x(t)$ has no zeros in $[c, \infty)$. Differentiation of (12) yields that $x(t)$ satisfies (1) for $t \geq c$ and $x'(t) > 0$ for $t > c$, i.e. (8) is not ture for $k = 1$ and $t > c$. Clearly $x(t)$ can be extended to a solution of (1) on $[0, \infty)$, and this solution is eventually positive but not strongly decreasing.

Remark. Taking $r_i(t) = 1$, $i = 1, 2, \cdots, n$, $f = p(t)\, x(t)$, $F = q(t)\, x(t)$ and n is odd, then Lovelady's results [7, Theorems 2 and 3] are special cases of our Theorems 1 and 2 respectively.

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