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On the positive bounded Solutions of linear delay higher order differential equations

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Equazioni differenziali ordinarie. — *On the positive bounded solutions of linear delay higher order differential equations* (*). Nota di LU–SAN CHEN e CHEH–CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dànno condizioni sufficienti perché l'equazione

$$L_n x(t) + (-1)^{n+1} a(t) x(g(t)) = 0$$

abbia soluzione positiva limitata.

1. INTRODUCTION

In this paper we consider the n-th order ($n > 1$) linear delay differential equation

$$L_n x(t) + (-1)^{n+1} a(t) x(g(t)) = 0, \tag{1}$$

where the differential operator L_n is defined by

$$L_0 x(t) = x(t), L_i x(t) = r_i(t) (L_{i-1} x(t))', \qquad i = 1, 2, \cdots, n,$$
$$r_n(t) = 1$$

and the functions $r_i(t)$ $(i = 1, \cdots, n-1)$ are positive at least on $[\tau, \infty)$, $\tau > 0$. Let n be an integer, $n > 1$, $a(t)$ be a positive continuous function on $[\tau, \infty)$ and let G be the set to which $g(t)$ belongs if and only if $g(t)$ is a non-negative, nondecreasing, unbounded continuous function on $[\tau, \infty)$ such that $g(t) \leq t$ whenever $t \geq \tau$. Let G^0 be the subset of G to which $g(t)$ belongs if and only if $g(t)$ is in G and $g(t) < t$ whenever $t > \tau$.

We give here some conditions to ensure that (1) has a positive bounded solutions. The technique used is an adaptation of that of Lovelady [1] which concerns the particular case $r_1(t) = r_2(t) = \cdots = r_{n-1}(t) = 1$. In what follows the term "solution" is always used only for such solutions $x(t)$ of (1) which are defined for all large t.

2. LEMMAS

To obtain our results we need the following two lemmas. The first of them is due to Lovelady [2], and the second is an improved version of another Lovelady's Lemma [1].

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LEMMA 1. *Let $y(t)$ be a positive bounded solution of*

$$L_n y(t) + (-1)^{n+1} a(t) y(t) = (-1)^n \varphi(t), \tag{2}$$

where $\varphi(t)$ is a positive continuous function on $[\tau, \infty)$. Let

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty, \qquad (i = 1, \cdots, n-1) \tag{C_1}$$

and

$$H[r_1, \cdots, r_{n-1}; a(s)] \tag{C_2}$$

$$\equiv \int^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} a(s)\, ds\, ds_{n-1} \cdots ds_1 = \infty .$$

If $k = 1, 2, \cdots, n$, then $u_k(t)$ is monotone, $\lim_{t\to\infty} u_k(t) = 0$ and there exists $t_b \geq \tau$ such that for $t \geq t_b$

$$u_k(t) \geq 0 \quad \text{if } k \text{ is odd,}$$

$$u_k(t) \leq 0 \quad \text{if } k \text{ is even,}$$

where

$$u_1(t) = y(t), u_2(t) = r_1(t) u_1'(t), \cdots, u_n(t) = r_{n-1}(t) u_{n-1}'(t).$$

LEMMA 2. *Suppose that the conditions (C_1) and (C_2) hold. Let $\psi(t)$ be a positive continuous function on $[\tau, \infty)$. Suppose also that $\psi(t) \leq \varphi(t)$ for $t \geq \tau$, and that there exists a positive bounded solution $y(t)$ of (2). Then there exists a positive bounded solution $x(t)$ of*

$$L_n x(t) + (-1)^{n+1} a(t) x(t) = (-1)^n \psi(t) \tag{3}$$

on $[\tau, \infty)$ such that

$$x(t) \leq y(t)$$

for $t \geq \tau$.

Proof. We define the following functions on $[\tau, \infty)$

$$u_1(t) = y(t), u_2(t) = r_1(t) u_1'(t), \cdots, u_n(t) = r_{n-1}(t) u_{n-1}'(t).$$

Then, by Lemma 1, we obtain for $k = 1, 2, \cdots, n$

$$\lim_{t\to\infty} u_k(t) = 0. \tag{4}$$

Integrating (2) $n-1$ times and using (4) we obtain

$$-u_1'(t) = \frac{1}{r_1(t)} H(r_2, \cdots, r_{n-1}; \varphi(s) + a(s) y(s)). \tag{5}$$

and

$$y(t) = u_1(t) = \mathrm{H}(r_1, \cdots, r_{n-1}; \varphi(s) + a(s) y(s)) \tag{6}$$

for $t \geq \tau$. It follows from (5), (6) and $\varphi(s) \geq \psi(s)$ that

$$-y'(t) \geq \frac{1}{r_1(t)} \mathrm{H}(r_2, \cdots, r_{n-1}; \psi(s) + a(s) y(s)) \tag{7}$$

and

$$y(t) \geq \mathrm{H}(r_1, \cdots, r_{n-1}; \psi(s) + a(s) y(s)) \tag{8}$$

for $t \geq \tau$. Consider the positive function sequence $\{z_k(t)\}_{k=1}^{\infty}$ which are defined on $[\tau, \infty)$ as follows:

$$z_1(t) = y(t)$$

$$z_{k+1}(t) = \mathrm{H}(r_1, \cdots, r_{n-1}; \psi(s) + a(s) z_k(s))$$

if $k \geq 1$. By an induction argument, we see easily that for $t \geq \tau$ and k is a positive integer

$$0 \leq z_{k+1}(t) \leq z_k(t) \leq y(t).$$

This and (7) imply that $\{z_k(t)\}_{k=1}^{\infty}$ is equicontinuous. Thus there exists a subsequence $\{z_{n_k}(t)\}_{k=1}^{\infty}$ of $\{z_k(t)\}_{k=1}^{\infty}$, which converges uniformly to $x(t)$. Clearly $x(t) \leq y(t)$ for $t \geq \tau$, and by the Dominated Convergence Theorem we get for $t \geq \tau$

$$x(t) = \mathrm{H}(r_1, \cdots, r_{n-1}; \psi(s) + a(s) x(s)). \tag{9}$$

Differentiating (9) yields (3) for $t \geq \tau$. This completes our proof.

Remark. From (7), (8), (9) and the facts that $x(t) \leq y(t)$ and $\psi(t) \leq \varphi(t)$ on $[\tau, \infty)$, we see that $-x'(t) \leq -y'(t)$ and $x'(t) < 0$ for $t \geq \tau$.

3. Main results

THEOREM 1. *Let the conditions* (C_1) *and* (C_2) *hold.*
Suppose that $g(t) \in G^0$ *and that*

$$L_n x(t) + (-1)^{n+1} a(t) x(t) = (-1)^n (t - g(t)) a(t) \tag{10}$$

has a positive bounded solution. Then (1) *has a positive bounded solution.*

Proof. Let $W_1(t)$ be a bounded positive solution of (10). Since $W_1(t) > 0$, $W_1'(t) \leq 0$, $W_1''(t) \geq 0$, we know that $W_1(\infty) = \lim_{t\to\infty} W_1(t)$ and $W_1'(\infty) = \lim_{t\to\infty} W_1'(t)$ both exist. Also, $W_1'(\infty) = 0$ for otherwise $W_1(\infty)$ and $W_1'(\infty)$ cannot both exist. Now, we find $t_0 \geq \tau$ such that $|W_1'(t)| \leq 1$ for $t \geq g(t_0)$.

Let $t_b > t_0$ and let $\lambda(t)$ and $\mu(t)$ be continuous nonnegative functions on $[t_0, \infty)$ such that $\lambda(t) + \mu(t) = 1$ for $t \geq t_0$ and $\lambda(t) = 1$, $\mu(t) = 0$ if $t \geq t_b$ and such that $\lambda(t) > 0$, $\mu(t) > 0$ if $t_0 \leq t \leq t_b$. Since $W_1'(t) \leq 0$, $W_1(t)$ is nonincreasing. Hence $W_1(g(t)) \geq W_1(t)$. If $t \geq t_0$ then

$W_1(g(t)) - W_1(t) = |W_1(g(t)) - W_1(t)| = |W_1'(\theta)(t - g(t))| \leq t - g(t)$ for some $\theta \in (g(t), t)$. Thus by Lemma 2, there is a bounded positive solution $W_2(t)$ on $[t_0, \infty)$ of

$$L_n W_2(t) + (-1)^{n+1} a(t) W_2(t)$$

$$= (-1)^n \mu(t)(t - g(t)) a(t) + (-1)^n \lambda(t) a(t) [W_1(g(t)) - W_1(t)]$$

with $0 \leq W_2(t) \leq W_1(t)$ and $-W_2'(t) \leq -W_1'(t)$ for $t \geq t_0$. Extend $W_2(t)$ to $[g(t_0), \infty)$ by requiring $W_2(t) = W_2(t_0)$ if $g(t_0) \leq t \leq t_0$. Integrating $-W_2'(t) \leq -W_1'(t)$ from $g(t)$ to t, we have

$$W_2(g(t)) - W_2(t) \leq W_1(g(t)) - W_1(t).$$

Hence Lemma 2 implies that there exists a bounded positive solution $W_3(t)$ on $[t_0, \infty)$ of

$$L_n W_3(t) + (-1)^{n+1} a(t) W_3(t)$$

$$= (-1)^n \mu(t)(t - g(t)) a(t) + (-1)^n \lambda(t) a(t) [W_2(g(t)) - W_2(t)]$$

with $0 \leq W_3(t) \leq W_2(t)$ and $-W_3'(t) \leq -W_2'(t)$ on $[t_0, \infty)$. Extend $W_3(t)$ to $[g(t_0), \infty)$ by requiring $W_3(t) = W_3(t_0)$ if $g(t_0) \leq t \leq t_0$. Continuing this way, we have a sequence $\{W_k(t)\}_{k=1}^{\infty}$ of positive nonincreasing functions such that

$$0 \leq W_{k+1}(t) \leq W_k(t) \leq W_1(t) \tag{11}$$

$$-W_{k+1}'(t) \leq -W_k'(t) \leq -W_1'(t) \tag{12}$$

and

$$L_n W_{k+1}(t) + (-1)^{n+1} a(t) W_{k+1}(t) \tag{13}$$

$$= (-1)^n \mu(t)(t - g(t)) a(t) + (-1)^n \lambda(t) a(t) [W_k(g(t)) - W_k(t)]$$

fot $t \geq t_0$, $k \geq 1$. By (11), $\{W_k(t)\}_{k=1}^{\infty}$ converges pointwise, and (12) says that the function sequence is equicontinuous, so $\{W_k(t)\}_{k=1}^{\infty}$ has a locally uniform limit, say $y(t)$. Now, (13) says that $\{L_n W_k(t)\}_{k=1}^{\infty}$ converges locally uniformly, so $L_n y(t)$ exists on $[t_0, \infty)$ and $L_n W_k(t) \to L_n y(t)$ locally uniformly and

$$L_n y(t) + (-1)^{n+1} a(t) y(t) \tag{14}$$

$$= (-1)^n \mu(t)(t - g(t)) a(t) + (-1)^n \lambda(t) a(t) [y(g(t)) - y(t)],$$

if $t \geq t_0$. For $t \geq t_b$, we have $\lambda(t) = 1$, $\mu(t) = 0$. Hence (14) gives (1) for $t \geq t_b$, so $y(t)$ is a solution of (1). Clearly $y(t)$ is bounded. Next, we shall prove that $y(t)$ is positive. Clearly $y(t)$ is nonnegative and nonincreasing, so if $T \geq t_0$ and $y(T) = 0$ then $y(t) = 0$ for $t \geq T$. Suppose $t_0 \leq T < t_b$ and $y(T) = 0$. Now, $y(t) = 0$ for $t \geq T$, so $L_n y(T) = 0$ and (14) is violated since $\mu(T)(T - g(T)) a(T) > 0$. Suppose $y(t) > 0$ on $[t_0, t_b)$, $y(t)$ has a zero and T is the first such zero, i.e., $T \geq t_b$, $y(t) > 0$ on $[t_0, T)$ and $y(T) = 0$. Now, $L_n y(t) = 0$ and since $g(T) < T$, $a(T) y(g(T)) > 0$; contradicting (1). Thus $y(t) > 0$ for $t \geq t_0$. Hence the proof is complete.

Remark. From the first inequality of (12), $-W'_{k+1}(t) \leq -W'_k(t)$ for $k \geq 1$ we have

$$-\int_{g(t)}^{t} W'_{k+1}(s)\, ds \leq -\int_{g(t)}^{t} W'_k(s)\, ds \quad \text{for} \quad k \geq 1,$$

that is

$$W_{k+1}(g(t)) - W_{k+1}(t) \leq W_k(g(t)) - W_k(t) \quad \text{for} \quad k \geq 1.$$

THEOREM 2. *Let $g(t)$, $h(t)$ be in G^0 and $g(t) \leq h(t)$ for $t \geq \tau$. Suppose that there exists a positive bounded solution of* (1). *If the conditions* (C_1) *and* (C_2) *hold, then there is a bounded positive solution of*

$$L_n W(t) + (-1)^{n+1} a(t) W(h(t)) = 0. \tag{15}$$

Proof. Let $y(t)$ be a bounded positive solution of (1). It follows from Lemma 1 that there exists a $t_0 \geq \tau$ such that for $t \geq t_0$ and $k = 1, 2, \cdots, n$

$$u_k(t) \geq 0 \quad \text{if } k \text{ is odd}$$

$$u_k(t) \leq 0 \quad \text{if } k \text{ is even}$$

and

$$\lim_{t \to \infty} u_k(t) = 0$$

where

$$u_1(t) = y(t), \; u_2(t) = r_1(t) u'_1(t), \cdots, u_n(t) = r_{n-1}(t) u'_{n-1}(t).$$

Let $t_b > t_0$ be such that $g(t_b) \geq t_0$. Define $\tilde{y}(t)$ by

$$\tilde{y}(t) = y(t_b) \quad \text{if} \quad t \leq t_b$$

$$\tilde{y}(t) = y(t) \quad \text{if} \quad t > t_b.$$

Let $x(t) = y(t) - \tilde{y}(t)$, then $x(t) > 0$ on $[g(t_b), t_b)$ and $x(t) = 0$ on $[t_b, \infty)$. Since, $y(t)$ is a solution of (1).

$$L_n \tilde{y}(t) + (-1)^{n+1} a(t) \tilde{y}(g(t)) = (-1)^n a(t) x(g(t))$$

for $t \geq t_0$. Thus for $t \geq t_b$

$$\tilde{y}(t) = \mathrm{H}(r_1, \cdots, r_{n-1}; a(s)[\tilde{y}(g(s)) + x(g(s))]). \tag{16}$$

Since $\tilde{y}(t)$ is nonincreasing, $\tilde{y}(g(t)) \geq \tilde{y}(h(t))$ for $t \geq t_b$. Hence (16) implies for $t \geq t_b$

$$\tilde{y}(t) \geq \mathrm{H}(r_1, \cdots, r_{n-1}; a(s)[\tilde{y}(h(s)) + x(g(s))]). \tag{17}$$

As in the proof of Theorem 1 we have a bounded nonnegative solution $\mathrm{W}(t)$ of

$$\mathrm{W}(t) = \mathrm{H}(r_1, \cdots, r_{n-1}; a(s)[\mathrm{W}(h(s)) + x(g(s))]), \tag{18}$$

or

$$\mathrm{L}_n \mathrm{W}(t) + (-1)^{n+1} a(t) \mathrm{W}(h(t)) = (-1)^n a(t) x(g(t)), \tag{19}$$

for $t \geq t_b$. The positivity of $x(t)$ on $[g(t_b), t_b)$ and the fact that $h(t) \in \mathrm{G}^0$, ensure as before that $\mathrm{W}(t)$ has no zeros.

If $\mathrm{T} > t_b$ and $g(\mathrm{T}) > t_b$, then (19) yields (15) for $t \geq \mathrm{T}$. Hence our proof is complete.

REFERENCES

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