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Michal Kisielewicz

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Analisi funzionale. — *Approximation theorem for set-valued functions.* Nota di MICHAŁ KISIELEWICZ, presentata (*) dal Corrisp. G. CIMMINO.

RIASSUNTO. — La presente Nota contiene la dimostrazione di un teorema di approssimazione per funzioni a valori insiemi compatti convessi. Si dimostra che ogni funzione $F(t, x)$ soddisfacente condizioni di tipo Carathéodory può approssimarsi con una localmente lipschitziana.

This Note is concerned with the approximation of set-valued mappings satisfying Carathéodory type conditions by means of locally Lipschitzian set-valued mappings. This theorem can be useful for proving existence theorems in the theory of differential equations for set-valued functions.

Let R^n denote the Euclidean n-space with the norm $\|\cdot\|$ and let (H, r) be the metric space of all non-empty compact convex subsets of R^n, where r is the metric given by the Hausdorff distance. For $A, B \in H$ and $\lambda \in R$ we define

$$(1) \qquad A + B = \{x + y : x \in A, y \in B\} \qquad \text{and} \qquad \lambda \cdot A = \{\lambda x : x \in A\}.$$

Let D be a measurable subset of R^m such that the Lebesgue measure $|D|$ of D satisfies $0 < |D| < \infty$. A function $F : D \to H$ is called measurable if for every $C \in H$ the set $\{t \in D : F(t) \cap C \neq \emptyset\}$ is Lebesgue measurable. A measurable function $F : D \to H$ is called Hukuhara integrable ([3]) if the single-valued function $g(t) = r(F(t), \{0\})$, where $0 \in R^n$, is Lebesgue integrable on D. The set of all Hukuhara integrable functions $F : D \to H$ will be denoted by X_D. It was proved in [2] that (X_D, Dist), where $\text{Dist}(F, G) = \int_D r(F(t), G(t))\, dt$, is a complete metric space. For $F \in X_D$ the Hukuhara integral of F on D will be denoted by $\int_D F(t)\, dt$. If R is the set of the real numbers and K a given set of vectors, together with the functions $+ : K \times K \to K$ and $\cdot : R \times K \to K$, K is called a quasilinear space over R, if and only if all the axioms for a linear space do hold, except (i) the distributivity of « · » over scalar addition and (ii) the existence of an inverse under « + ». We shall call the metric space (Y, d) a quasinormed space if Y is a quasilinear space over R and $d(x + u, y + v) \leq d(u, v) + d(x, y)$, $(\lambda + \gamma) \cdot x = \lambda \cdot x + \gamma \cdot x$ and $d(\lambda \cdot x, \lambda \cdot y) = \lambda d(x, y)$ for $x, y, u, v \in Y$ and $\lambda, \gamma \geq 0$. In the sequel we will denote by $\hat{0}$ the element of Y of the form

(*) Nella seduta del 13 novembre 1976.

$0 \cdot x$ for $x \in Y$. It follows from [1] and [2] that (H, r) together with the functions « + » and « · » defined by (1) is a quasinormed space. Hence it follows that (X_D, Dist), together with the functions $\oplus$ and $\odot$ defined by

$$(2) \qquad (F \oplus G)(t) = F(t) + G(t) \quad \text{and} \quad (\lambda \odot F)(t) = \lambda \cdot F(t)$$

for $F, G \in X_D$, $\lambda \in R$ and $t \in D$ is a quasinormed space, too.

Let (X, ρ) and (Y, d) be metric spaces and let $U \subset X$ be an open set. We will say $f: U \to Y$ is locally Lipschitzian if for each $x \in U$ there are an open set Ω_x with $x \in \Omega_x \subset U$ and a $L_x > 0$ such that $d(f(x_1), f(x_2)) \leq L_x \rho(x_1, x_2)$ for all $x_1, x_2 \in \Omega_x$. A function f will be said bounded in U if there is a number $M > 0$ such that $d(f(x), \hat{0}) \leq M$ for every $x \in U$. In similar way as in [4] we prove

THEOREM 1. *Let* (X, ρ) *and* (Y, d) *be respectively a metric and a quasinormed space and let* $U \subset X$ *be open. Let* $f: U \to Y$ *be continuous and bounded on* U. *Then for every* $\varepsilon > 0$ *there exists a locally Lipschitzian and bounded function* $f^\varepsilon: U \to Y$ *such that* $d(f^\varepsilon(x), f(x)) < \varepsilon$ *for all* $x \in U$.

Proof. For fixed $x \in U$ put $N(\varepsilon, x) = \{y \in U : \rho(x, y) < 1$ and $d(f(x), f(y)) < \varepsilon\}$. Then $U = \bigcup_{x \in U} N(\varepsilon/2, x)$. Since every metric space is paracompact there is a locally finite refinement $\{Q_\alpha\}_{\alpha \in A}$ of $\{N(\varepsilon/2, x) : x \in U\}$, where Q_α is nonempty and open. Define $\mu_\alpha : X \to [0, \infty)$ and $p_\alpha : X \to [0, 1]$ for all $\alpha \in A$ by

$$\mu_\alpha(x) = \begin{cases} 0 & \text{if } x \notin Q_\alpha \\ D(x, \partial Q_\alpha) & \text{if } x \in Q_\alpha \end{cases} \qquad \text{and} \qquad p_\alpha(x) = \mu_\alpha(x) \left[\sum_{\beta \in A} \mu_\beta(x)\right]^{-1},$$

where $D(x, \partial Q_\alpha) = \inf\{\rho(x, z) : z \in \partial Q_\alpha\}$.

It is not difficult to see that μ_α is Lipschitzian with constant 1. Hence it follows that p_α is locally Lipschitzian in U. Let $\{x_\alpha\}_{\alpha \in A}$ be a subset of U such that $x_\alpha \in Q_\alpha$ for $\alpha \in A$. Let us define $f^\varepsilon(x) = \sum_{\alpha \in A} p_\alpha(x) \cdot f(x_\alpha)$ for $x \in U$. Since $\{Q_\alpha\}_{\alpha \in A}$ is locally finite, then for every $x \in U$ there is an open set Ω_x with $x \in \Omega_x \subset U$ such that

$$d(f^\varepsilon(x_1), f^\varepsilon(x_2)) \leq \sum_{\alpha \in A} d(p_\alpha(x_1) \cdot f(x_\alpha), p_\alpha(x_2) \cdot f(x_\alpha)) \leq$$

$$\leq \sum_{\alpha \in A} [p_\alpha(x_1) - p_\alpha(x_2)] \cdot$$

$$\cdot d\left(\left[1 + \frac{p_\alpha(x_2)}{p_\alpha(x_1) - p_\alpha(x_2)}\right] \cdot f(x_\alpha), \frac{p_\alpha(x_2)}{p_\alpha(x_1) - p_\alpha(x_2)} \cdot f(x_\alpha)\right) \leq$$

$$\leq \sum_{\alpha \in A} [p_\alpha(x_1) - p_\alpha(x_2)] \, d(f(x_\alpha), \hat{0})$$

for $x_1, x_2 \in \Omega_x$ and $p_\alpha(x_1) > p_\alpha(x_2)$. Similarly, for $p_\alpha(x_1) < p_\alpha(x_2)$ we obtain

$$d(f^\varepsilon(x_1), f^\varepsilon(x_2)) \leq \sum_{\alpha\in A} [p_\alpha(x_2) - p(x_1)] \cdot$$

$$\cdot d\left(\frac{p_\alpha(x_1)}{p_\alpha(x_2) - p_\alpha(x_1)} \cdot f(x), \left[\frac{p_\alpha(x_1)}{p_\alpha(x_2) - p_\alpha(x_1)} + 1\right] \cdot f(x_\alpha)\right) \leq$$

$$\leq \sum_{\alpha\in A} [p_\alpha(x_2) - p_\alpha(x_1)] \, d(f(x_\alpha), \hat{o}).$$

Then for $x_1, x_2 \in \Omega_x$ and $p_\alpha(x_1) \neq p_2(x_2)$ we may deduce the existence of a number $L_x > 0$ such that $d(f^\varepsilon(x_1), f^\varepsilon(x_2)) \leq L_x \rho(x_1, x_2)$. Hence it follows that f is locally Lipschitzian. Similarly, for $x \in U$ we obtain $d(f^\varepsilon(x), \hat{o}) \leq M$. Moreover, by the definition of $\{Q_\alpha\}_{\alpha\in A}$ for every $x \in U$ there is $\alpha_x \in A$ so that $x \in Q_{\alpha_x}$. Then for $\alpha_x \in A$ there is $\tilde{x} \in U$ such that $Q_{\alpha_x} \subset N(\varepsilon/2, \tilde{x})$. Therefore for $x \in U$ there is $\tilde{x} \in U$ such that $x \in Q_{\alpha_x} \subset N(\varepsilon/2, \tilde{x})$. Since $p_\alpha(x) = 0$ for $\alpha \neq \alpha_x$ then

$$d(f(x), f^\varepsilon(x)) = d\left(\sum_{\alpha\in A} p_\alpha(x) \cdot f(x), \sum_{\alpha\in A} p_\alpha(x) \cdot f(x_\alpha)\right) \leq$$

$$\leq \sum_{\alpha\in A} p_\alpha(x) \, d(f(x), f(x_\alpha)) \leq \sum_{\alpha_x\in A} p_{\alpha_x}(x) \, [d(f(x), f(\tilde{x})) + d(f(\tilde{x}), f(x_{\alpha_x}))] \leq$$

$$\leq \varepsilon \sum_{\alpha\in A} p_\alpha(x) = \varepsilon.$$

This completes the proof.

We shall say that $F: D \times U \to H$ satisfies Carathéodory type conditions if $F(\cdot, x)$ is measurable in $t \in D$, $F(t, \cdot)$ is continuous in $x \in U$ and there is a Lebesgue integrable function $m: D \to R$ so that $r(F(t, x), \{0\}) \leq m(t)$ for $x \in U$ and a.e. $t \in D$. Here $F(\cdot, x)$ denotes the function of the form $D \ni t \to F(t, x)$ for fixed $x \in U$. Now we can prove the following theorem:

THEOREM 2. *Let* (X, ρ) *be a metric space and let* (H, r) *denote the space of all nonempty compact convex subsets of* R^n *with the Hausdorff metric* r. *Suppose* U *is an open set of* X *and* D *is a measurable subset of* R^m *such that* $0 < |D| < \infty$. *Let* $F: D \times U \to H$ *satisfy Carathéodory type conditions. Then for every* $\eta > 0$ *there is a mapping* $F^\eta: D \times U \to H$ *satisfying Carathéodory type conditions with following property: for every* $x \in U$ *there are an open set* Ω_x *with* $x \in \Omega_x \subset U$ *and a number* $L_x > 0$ *such that* $\text{Dist}(F(\cdot, x), F^\eta(\cdot, x)) < \eta$ *for each* $x \in U$ *and* $\int_D r(F^\eta(t, x_1), F^\eta(t, x_2)) \, dt \leq L_x \rho(x_1, x_2)$ *for* $x_1, x_2 \in \Omega_x$.

Proof. Let $f(x) = F(\cdot, x)$ for fixed $x \in U$. We have $f(x) \in X_D$ for every $x \in U$. Therefore $f: U \to X_D$. It is not difficult to verify that f satisfies the assumptions of Theorem 1. Then for $\varepsilon = 1/n$ there is a locally Lipschitzian function $f_n: U \to X_D$ so that $\text{Dist}(f_n(x), f(x)) < 1/n$ for $x \in U$. There is a subsequence, say $\{f_k\}$ of $\{f_n\}$ such that $r(f_k(x)(t), f(x)(t)) \to 0$ as $k \to \infty$ for a.e. $t \in D$ and uniformly with respect to $x \in U$. Then for every $\eta > 0$

there is a positive integer k_η such that $r(f_{k_\eta}(x)(t), f(x)(t)) < \eta$ for a.e. $t \in D$ and $x \in U$. Let $F^\eta(t, x) = f_{k_\eta}(x)(t)$. It is easy to see that F^η satisfies Carathéodory type conditions. For $x \in U$ we have $\mathrm{Dist}(F^\eta(\cdot, x), F(\cdot, x)) < \eta$. Furthermore for every $x \in U$ there are an open set Ω_x with $x \in \Omega_x \subset U$ and a number $L_x > 0$ such that $\mathrm{Dist}(f_{k_\eta}(x_1), f_{k_\eta}(x_2)) \le L_x \rho(x_1, x_2)$. Therefore for $x_1, x_2 \in \Omega_x$ we have

$$\int_D r(F^\eta(t, x_1), F^\eta(t, x_2))\, dt \le L_x \rho(x_1, x_2).$$

This completes the proof.

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