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Fixed point theorems for quasi-nonexpansive mappings

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Analisi funzionale. — *Fixed point theorems for quasi-nonexpansive mappings.* Nota di KANHAYA L. SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabiliti vari teoremi del punto fisso per applicazioni quasi non espansive in spazi di Banach. Si dimostra inoltre che in uno spazio di Hilbert ogni applicazione quasi non espansiva risulta ragionevolmente errabonda ed asintoticamente regolare in sensi qui definiti. Si ottiene infine un teorema di convergenza debole per le iterate di una applicazione quasi non espansiva.

INTRODUCTION

Recently many authors (Kirk, Edelstein, Gobel, Browder, Singh, etc.) have proved fixed point theorems for nonexpansive operators mapping a closed bounded and convex subset of a Banach space into itself. The main purpose of this paper is to establish the existence of fixed points for operators mapping a closed bounded convex subset of a Banach space into itself which instead of being nonexpansive are quasi-nonexpansive. The concept of quasi-nonexpansive mapping was first communicated by the author to R. K. Yadav ("Banaras Math. Journal", 1969). Further this kind of mapping was studied by Soardi, Reich, the present author and others.

DEFINITION 1.1. Let C be a closed, bounded and convex subset of a Banach space X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\| Tx - Ty \| \leq \| x - y \| \qquad \text{for all } x, y \text{ in C.}$$

DEFINITION 1.2. A mapping $T : C \to C$ is said to be quasi-nonexpansive if

$$\| Tx - Ty \| \leq (\| x - Tx \| + \| x - y \| + \| y - Ty \|)/3 \qquad \text{for all } x, y \text{ in C.}$$

The following example shows that there are quasi-nonexpansive mappings which are not necessarily nonexpansive.

EXAMPLE 1.1. Let $X = [0, 1]$ and let $Tx = x/3$ for $0 \leq x < 1$ and $T(1) = 1/6$. Then T is quasi-nonexpansive but it is not nonexpansive.

THEOREM 1.1. *Let* C *be a weakly compact convex subset of a normed linear space* X. *Let* $T : C \to C$ *be a quasi-nonexpansive mapping. Furthermore assume* T *is continuous, then the set* $\{\| x - Tx \| / x \text{ in } C\}$ *has a smallest number.*

Proof. Let $r > 0$. Define the set C_r by $C_r = \{y \text{ in } C / \| y - Ty \| \leq r\}$. Let $D = \{r \text{ in } [0, \infty)/C_r \neq \varnothing\}$. Since C is bounded, $D \neq \varnothing$. It is enough to show that D has a smallest element or equivalently $\cap C_r \neq \varnothing$. For each

(*) Nella seduta del 13 novembre 1976.

r in D, let C_{1r} be the closed convex hull of $T(C_r)$. Since for any r, s in D with $r < s$, $C_r \subseteq C_s$, the family $F = \{C_{1r}/r \text{ in } D\}$ has a finite intersection property. Since each C_{1r} is closed and convex, it is weakly closed. Therefore by the weak compactness of C we conclude that F has nonempty intersection. Thus it is enough to show that $C_{1r} \subseteq C_r$ for each r in D. Let r in D, y in C_{1r}. Then there exists $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $[0, 1]$ and $y_1, y_2, \cdots, y_n$ in C such that $y = \Sigma\alpha_i T(y_i)$, $\alpha_i \geq 0$, $\Sigma\alpha_i = 1$. Thus

$$\|y - Ty\| = \|\Sigma\alpha_i T(y_i) - Ty\| \leq \Sigma\alpha_i \|T(y_i) - Ty\|$$
$$\leq \Sigma\alpha_i \{\|T(y_i) - y_i\| + \|y - y_i\| + \|y - Ty\|\}/3$$

or

$$\|y - Ty\| \leq \Sigma\alpha_i \{\|T(y_i) - y_i\| + \|y - y_i\|\}/3 + \Sigma\alpha_i \|y - Ty\|/3$$
$$\leq \Sigma\alpha_i \{\|T(y_i) - y_i\| + \|y - y_i\|\}/3 + \|y - Ty\|/3 .$$

Therefore [since $y = \Sigma\alpha_i T(y_i)$ and $y_i \in C_r$]:

$$2\|y - Ty\|/3 \leq \Sigma\alpha_i \|T(y_i) - y_i\|/3 + \Sigma\alpha_i \|T(y_i) - y_i\|/3$$
$$\leq 2/3\, \Sigma\alpha_i \|T(y_i) - y_i\| \leq 2/3\, \Sigma\alpha_i r = 2/3\, r .$$

Thus $\|y - Ty\| \leq r$; and this implies that $C_{1r} \subseteq C_r$.

THEOREM 1.2. *Let* X *be a normed linear space. Let* C *be a weakly compact convex subset of* X *and* $T : C \to C$ *be continuous quasi-nonexpansive. If* T *does not have a fixed point, then there exists a* T*-invariant closed convex subset of* X *such that* $\delta(H) > 0$ *and* $\|x - Tx\| = \delta(H)$ *for all* x *in* H, *where* $\delta(H)$ *is the diameter of* H. *Moreover if for any closed convex* T*-invariant subset* H *of* X *with* $\delta(H) > 0$ *there exists an* x *in* H *such that* $\|x - y\| < \delta(H)$, y *in* H, *then* T *has a unique fixed point.*

Proof. By Zorn's lemma there exists a nonempty T-invariant minimal closed convex subset D of C. By Theorem 1.1, there exists x in D such that

$$\|x - Tx\| = \inf\{\|y - Ty\|/y \text{ in } D\} .$$

Let $\alpha_0 = \|x - Tx\|$. By hypothesis $\alpha_0 > 0$. Using the notations of Theorem 1.1 we have $C_{1\alpha_0} \subseteq C_{\alpha_0}$,

$$T(C_{1\alpha_0}) \subseteq T(C_{\alpha_0}) \subseteq \overline{\text{co}}\, T(C_{\alpha_0}) = C_{1\alpha_0},$$

where $\overline{\text{co}}$ designate the convex closure.

By minimality of D we infer that $D = C_{1\alpha_0}$ and therefore $C_{\alpha_0} = D$. By the choice of α_0, $\|y - Ty\| = \alpha_0$ for all y in D. Now for any x in C_{α_0},

$$\|Tx - T^2 x\| \leq 1/3\, \{\|x - Tx\| + \|x - Tx\| + \|Tx - T^2 x\|\}$$

or

$$\|Tx - T^2 x\| \leq \|x - Tx\| \leq \alpha_0 .$$

Thus we conclude that $T(C_{\alpha_0}) \subseteq C_{\alpha_0}$, hence $\delta(TC_{\alpha_0}) \leq \delta(C_{\alpha_0}) = \alpha_0$. Therefore $\delta(D) = \alpha_0$. Hence $\|y - Ty\| = \delta(D)$ for all y in D.

The proof of second half follows from the proof of first part.

THEOREM 1.3. *Let* C *be a weakly compact convex subset of a Banach space* X. *Let* T *be a continuous quasi-nonexpansive self map of* C. *Suppose that for any closed convex subset* D *of* C *with* $T(D) \subset D$ *and* $\delta(D) > 0$

$$\inf \{\| y - Ty \| / y \text{ in } D\} < \delta(D) .$$

Then T *has a unique fixed point.*

Proof. By Zorn's lemma, there exists a minimal nonempty weakly compact convex H of C such that $T(H) \subset H$. Let x in H, $r = \| x - Tx \|$. Consider $W = \{y \text{ in } H / \| y - Ty \| \leq r\}$.

Since x in H, $W \neq \varnothing$. Since for any z in W

$$\| T^2 z - Tz \| \leq 1/3 \{\| z - Tz \| + \| z - Tz \| + \| T^2 z - Tz \|\}$$

or

$$\| T^2 z - Tz \| \leq \| z - Tz \| \leq r .$$

Thus we conclude that $T(W) \subset W$. Let V be the closure of convex hull of $T(W)$. We claim that $V \subset W$. Let u in V and $\varepsilon > 0$, then there exist $v_1, v_2, \cdots, v_n$ in W and $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $[0, 1]$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and

$$\left\| u - \sum_{i=1}^{n} \alpha_i Tv_i \right\| < \varepsilon .$$

Thus

$$\| u - Tu \| \leq \left\| u - \sum_{i=1}^{n} \alpha_i Tv_i \right\| + \left\| \sum_{i=1}^{n} \alpha_i Tv_i - Tu \right\| < \varepsilon + \sum_{i=1}^{n} \alpha_i \| Tv_i - Tu \| .$$

An argument similar to the proof of Theorem 1.1 shows that $\| u - Tu \| \leq \leq 2\varepsilon + r$. Since ε was arbitrary chosen, $\| u - Tu \| \leq r$, i.e. u in W. Thus $V \subset W$. So

$$T(V) \subset T(W) \subset V .$$

By minimality of H, $V = H$. So $W = H$. Since x is arbitrary chosen it follows from $W = H$ that $\| z - Tz \| = r$ for all z in H.

Construct D (take $H = D$) as above. Suppose $\delta(D) > 0$, then by hypothesis there exists x in D such that $r = \| x - Tx \| < \delta(D)$.

Construct V as above. Then $V = W = D$ and $r = \| y - Ty \|$ for each y in D. Let p in D. Then

$$\| T^2 p - Tp \| \leq \| p - Tp \| \leq r .$$

Thus $T(D) \subset D$, therefore $\delta(T(D)) \leq \delta(D) = r$. Thus $\delta(D) = \delta(V) = = \delta(T(W)) \leq r < \delta(D)$, a contradiction.

DEFINITION 2.1. Let H be a Hilbert space. Let C be a closed, convex subset of H. A mapping $T : C \to C$ is said to be a *reasonable wanderer* in C if starting at any x_0 in C, its successive steps $x_n = T^n x_0$ $(n = 1, 2, 3, \cdots)$ are such that the sum of squares of their lengths is finite, i.e.

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 < \infty .$$

THEOREM 2.1. *Let* H *be a Hilbert space. Let* C *be a closed convex subset of* H. *Let* $T : C \to C$ *be a quasi-nonexpansive mapping with nonempty fixed point set* F. *If* $T_\lambda = \lambda I + (1-\lambda) T$ *for any given* λ *with* $0 < \lambda < 1$, *then* T_λ *is a reasonable wanderer from* C *into* C *with the same fixed points as* T.

Proof. It is obvious that T and T_λ have the same fixed points. It remains to show that T is a reasonable wanderer. For any x in C, set $x_n = T_\lambda^n x$ and let y be a fixed point of T and hence of T_λ. Then

$$(1) \qquad x_{n+1} - y = \lambda x_n + (1-\lambda) T x_n - y = T_\lambda x_n - y = \lambda (x_n - y) + \\ + (1-\lambda)(T x_n - y) .$$

On the other hand, for any constant β,

$$(2) \qquad \beta (x_n - T x_n) = \beta (x_n - y) - \beta (T x_n - y) .$$

Let us first observe that

$$(3) \qquad \| T x_n - y \| = \| T x_n - T y \| \leq \\ \leq \{\| x_n - T x_n \| + \| x_n - y \| + \| y - T y \|\}/3 \\ = \{\| x_n - T x_n \| + \| x_n - y \|\}/3 .$$

Now

$$(4) \qquad \| x_n - T x_n \| \leq \| x_n - y \| + \| y - T x_n \| .$$

Thus substitution from (4) into (3) gives

$$(5) \qquad \| T x_n - y \| \leq \| x_n - y \| .$$

Since from (1) we have

$$(6) \qquad \| x_{n+1} - y \|^2 = \lambda^2 \| x_n - y \|^2 + (1-\lambda)^2 \| T x_n - y \|^2 + \\ + 2 \lambda (1-\lambda) (T x_n - y , x_n - y) ,$$

and from (2) we get

$$(7) \qquad \beta^2 \| x_n - T x_n \|^2 = \beta^2 \| x_n - y \|^2 + \\ + \beta^2 \| T x_n - y \|^2 - 2 \beta^2 (T x_n - y , x_n - y) .$$

Adding corresponding sides of (6) and (7) we obtain

$$(8) \qquad \| x_{n+1} - y \|^2 + \beta^2 \| x_n + T x_n \|^2 \\ = (\lambda^2 + \beta^2) \| x_n - y \|^2 + \{(1-\lambda)^2 + \beta^2\} \| T x_n - y \|^2 + \\ + 2 \{\lambda (1-\lambda) - \beta^2\} (T x_n - y , x_n - y) .$$

Using (5) we can write (8) as

$$(9) \qquad \| x_{n+1} - y \|^2 + \beta^2 \| x_n - T x_n \|^2 \leq \\ \{\lambda^2 + \beta^2 + (1-\lambda)^2 + \beta^2\} \| x_n - y \|^2 + 2 \{\lambda (1-\lambda) - \beta^2\} (T x_n - y , x_n - y) .$$

If we assume that β is such that $\beta^2 \leq \lambda(1-\lambda)$, then using the Cauchy-Schwarz inequality and (5), we obtain from (9).

$$(10) \qquad \|x_{n+1}-y\|^2+\beta^2\|x_n-Tx_n\|^2 \leq$$
$$(\lambda^2+\beta^2+1-2\lambda+\lambda^2+\beta^2+2\lambda-2\lambda^2-2\beta^2)\|x_n-y\|^2=\|x_n-y\|^2 .$$

Letting $\beta^2=\lambda(1-\lambda)>0$ and summing up from $n=0$ to $n=N$, we obtain

$$(11) \qquad \lambda(1-\lambda)\sum_{n=0}^{N}\|x_n-Tx_n\|^2 \leq \sum_{n=0}^{N}\{\|x_n-y\|^2-\|x_{n+1}-y\|^2\}$$
$$=\|x_0-y\|^2-\|x_{N+1}-y\|^2 \leq \|x_0-y\|^2 .$$

Hence $\sum_{n=0}^{\infty}\|x_n-Tx_n\|^2<\infty$. Since $x_{n+1}-x_n=\lambda x_n+(1-\lambda)Tx_n-x_n=$ $=(1-\lambda)(Tx_n-x_n)$, from (11) we obtain

$$\lambda(1-\lambda)\sum_{n=0}^{\infty} 1/(1-\lambda)^2\|x_{n+1}-x_n\|^2 \leq \|x_0-y\|^2$$

or

$$\sum_{n=0}^{\infty}\|x_{n+1}-x_n\|^2 \leq (1-\lambda)/\lambda\|x_0-y\|^2 .$$

Therefore T_λ is a reasonable wanderer in C.

DEFINITION 2.2. Let H be a Hilbert space and C be a closed convex subset of H. A mapping $T: C \to C$ is called *asymptotically regular* at x if and only if $\|T^n x - T^{n+1} x\| \to 0$ as $n \to \infty$.

COROLLARY 2.1. *If* T *is quasi-nonexpansive and the set* F *of fixed points of* T *in* C *is nonempty and if* $T_\lambda=\lambda I+(1-\lambda)T$ *for a given* λ *with* $0<\lambda<1$, *then* T_λ *has the same fixed points as* T, *and* T_λ *is asymptotically regular.*

Remark 2.1. A Theorem similar to Theorem 2.1 for nonexpansive mappings is given by Browder and Petryshyn (3); unfortunately, the Theorem is not stated correctly there. The mapping T_λ should be defined by $T_\lambda=\lambda I+(1-\lambda)T$ instead of $T_\lambda=I+(1-\lambda)T$.

THEOREM 3.1. *Let* X *be a Banach space and* x_0 *be any point of* X. *Let* T *be a mapping of* X *into itself satisfying*

$$(1) \qquad \|Tx-Ty\| \leq 1/3\{\|x-Tx\|+\|x-y\|+\|y-Ty\|\}$$

x, y *in* X. *Suppose the sequence* $\{x_n\}$, *where* $x_{n+1}=(x_n+Tx_n)/2$ *converges to* u. *Then* u *is a fixed point of* T.

Proof. Define the mapping F as

$$Fx=(x+Tx)/2 .$$

Then F maps X into itself and the sequence $\{x_n\}$ becomes the sequence of iterates of x_0 by F. Now if x, y in X, then

$$\begin{aligned}\|Fx - Fy\| &= 1/2\,\|x - y + Tx - Ty\| \leq 1/2\,\{\|x - y\| + \|Tx - Ty\|\}\\ &\leq 1/2\,\{\|x - y\| + 1/3\,(\|x - Tx\| + \|x - y\| + \|y - Ty\|)\}\\ &= 1/3\,\{2\,\|x - y\| + 1/2\,(\|x - Tx\| + \|y - Ty\|)\}\\ &= 1/3\,\{2\,\|x - y\| + \|x - Fx\| + \|y - Fy\|\}\,.\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - Fu\| = \|Fx_n - Fu\| &\leq 2\,\|x_n - u\|/3 + \{\|x_n - Fx_n\| + \|u - Fu\|\}/3\\ &\leq 2\,\|x_n - u\|/3 + \{\|x_n - u\| + \|u - x_{n+1}\| + \|u - x_{n+1}\|\\ &+ \|u - x_{n+1}\| + \|x_{n+1} - Fu\|\}/3\end{aligned}$$

or

$$\|x_{n+1} - Fu\| \leq 3\,\|x_n - u\|/2 + \|u - x_{n+1}\|\,.$$

Since $x_n \to u$ as $n \to \infty$, thus we have $\|x_{n+1} - Fu\| \to 0$ as $n \to \infty$ hence $u = Fu$. So $u = Fu = (u + Tu)/2$, which implies $u = Tu$.

THEOREM 3.2. *Let $\{f_n\}$ be a sequence of elements in a Banach space. Let g_n be the unique solution of the equation $h - T(h) = f_n$, where T is a mapping of X into itself such that*

(1) $\|Tx - Ty\| \leq 1/3\,\{\|x - Tx\| + \|x - y\| + \|y - Ty\|\}$,

x, y in X. If $\|f_n\| \to 0$ as $n \to \infty$, then the sequence $\{g_n\}$ converges to the solution of the equation $h = T(h)$.

Proof. Using condition (1) we will show that $\{g_n\}$ is a Cauchy sequence.

(2) $\|g_n - g_m\| \leq \{\|g_n - Tg_n\| + \|Tg_n - Tg_m\| + \|Tg_m - g_m\|\,.$

Now

(3) $\|Tg_n - Tg_m\| \leq \{\|g_n - Tg_n\| + \|g_n - g_m\| + \|g_m - Tg_m\|\}/3\,.$

Hence using (3) we can write (2) as

$$2\,\|g_n - g_m\|/3 \leq 4\,\{\|g_n - Tg_n\| + \|g_m - Tg_m\|\}/3$$

or

$$\|g_n - g_m\| \leq 2\,\{\|g_n - Tg_n\| + \|g_m - Tg_m\|\} \leq 2\,(\|f_n\| + \|f_m\|)\,.$$

It follows, therefore, that $\{g_n\}$ is a Cauchy sequence in X. Hence it converges, say to g in X. Also

$$\begin{aligned}\|g - Tg\| &\leq \|g - g_n\| + \|g_n - Tg_n\| + \|Tg_n - Tg\|\\ &\leq \|g - g_n\| + \|g_n - Tg_n\| + \{\|g_n - Tg_n\| + \|g - g_n\| +\\ &+ \|g - Tg\|\}/3\,,\end{aligned}$$

or

$$\|g - Tg\| \leq 2\,\{\|g - g_n\| + \|g_n - Tg_n\|\}\,,$$

for arbitrary positive integer n. Hence taking the limit as $n \to \infty$ we get $\| g - Tg \| \to 0$, or $g = Tg$.

THEOREM 3.3. *Let* X *be a uniformly convex Banach space. Let* C *be a closed bounded convex subset of* X. *Let* $T: C \to C$ *be a quasi-nonexpansive mapping. If* x *in* C *and* c *is the asymptotic center of* $\{T^n x / n = 0, 1, 2, \cdots\}$ *then* $r(c) = \inf \{\| T^n x - c \| / n = 0, 1, 2, \cdots\} \le \inf \{\| T^n x - y \| / n = 0, 1, 2, \cdots\}$ *for each fixed point* y *of* T.

Proof. From $y = Ty$ it follows that

$$
\begin{aligned}
(1) \qquad \| T^{n+1} x - y \| &= \| T^{n+1} x - Ty \| \\
&\le \{\| T^n x - T^{n+1} x \| + \| T^n x - y \| + y - Ty \|\}/3 \\
&= \{\| T^n x - T^{n+1} x \| + \| T^n x - y \|\}/3 .
\end{aligned}
$$

Now

$$
(2) \qquad \| T^n x - T^{n+1} x \| \le \| T^n x - y \| + \| y - T^{n+1} x \| .
$$

Using (2) we can write (1) as

$$
(3) \qquad \| T^{n+1} x - y \| \le \| T^n x - y \| .
$$

From (3) we conclude $r_n(y) = \| T^n x - y \|$ and $r(y) = \inf \{\| T^n x - y \| / n = 0, 1, 2, \cdots\}$. The conclusion of Theorem 3.3 now follows from (6) of (7).

Before we state and prove our next theorem we need the following Lemmas:

LEMMA 4.1. (3, Proposition 1.4, pp. 32). Let X be a topological space and C be a compact subset of it. Let $g: X \to R$ be a lower semicontinuous function in X. Then there exists x_0 in C such that $g(x_0) = \inf_{x \in C} g(x)$.

LEMMA 4.2. (3, Proposition 2.5, pp. 53). Let X be a Banach space and g a convex continuous real-valued function on X. Then g is weakly lower semicontinuous.

LEMMA 4.3. *Let* H *be a Hilbert space and* C *be convex subset of* H. *Let* $T: C \to C$ *satisfy the following condition*:

$$
\| Tx - Ty \| \le a \| x - y \| + b (\| x - Tx \| + \| y - Ty \|)
$$

for all x, y *in* C *and* $a + 2b \le 1$. *Suppose* F, *the fixed point set of* T *in* C, *is nonempty*; *then* F *is convex*.

Proof. Let x, y be any two points of F, and $z = tx + (1 - t) y$, $0 < t < 1$. Since C is convex therefore z belongs to C. We claim that z belongs to F. Now

$$
\begin{aligned}
\| Tz - x \| &= \| Tz - Tx \| \le a \| z - x \| + b (\| x - Tx \| + \| z - Tz \|) \\
&= a \| z - x \| + b \| z - Tz \| \le a \| z - x \| + b (\| z - x \| + \| x - Tz \|) .
\end{aligned}
$$

Thus

$$\| Tz - x \| \leq (a + b) \| z - x \| / (\| z - x \| / (1 - b)) (1 - b) \leq \| z - x \| .$$

Similarly

$$\| Tz - y \| \leq \| z - y \| .$$

Now

$$z - x = tx + (1 - t) y - x = -(1 - t)(x - y) ,$$

and

$$z - y = tx + (1 - t) y - y = t (x - y) .$$

Thus

$$\| x - y \| \leq \| x - Tz \| + \| Tz - y \| \leq \| z - x \| + \| z - y \|$$

$$\leq (1 - t) \| x - y \| + t \| x - y \| = \| x - y \| .$$

Hence $\| x - Tz \| + \| Tz - y \| = \| x - Tz + Tz - y \|$. If $x - Tz = 0$, then $\| Tz - y \| = \| x - y \| \leq \| z - y \| = t \| x - y \|$, whence $t \geq 1$, which is not true. Similarly $Tz - y = 0$ implies $1 - t \geq 1$, whence $t \leq 0$, which is not true. Since H is strictly convex, therefore there exists $m > 0$ such that $Tz - x = m (y - Tz)$, whence $Tz = (1 - n) x + ny$, where $n = m/(1 + m)$. We have $Tz - x = n (y - x)$ and so $n \| y - x \| = \| Tz - x \| \leq \| z - x \| = t \| y - x \|$, which gives $n \leq t$. Using $Tz - y = (1 - n)(x - y)$, a similar argument gives $n \geq t$. Thus $n = t$ and so $Tz = (1 - t) x + ty = z$, i.e. z belongs to F.

THEOREM 4.1. *Let* H *be a Hilbert space and* T *a continuous asymptotically regular mapping of* H *into itself satisfying the condition* $\| Tx - Ty \| \leq a \| x - y \| + b (\| x - Tx \| + \| y - Ty \|)$ *for all* x, y *in* H *and* $a + 2b \leq 1$. *Suppose that the set* F *of fixed points of* T *is non-empty. Then, for each point* x_0 *in* H, *the sequence* $\{T^n x_0\}$ *converges weakly to a point of* F; *provided* I — T *is demiclosed.*

Proof. Since by assumption F is non-empty, therefore it follows that a ball B about some fixed point and containing x_0 is mapped into itself by T; consequently B contains the sequence of iterates $T^n x_0$. Thus without loss of generality we can restrict ourselves to a mapping of a ball into itself. By Lemma 4.3 it follows that F is convex, and by the continuity of T we get the closedness of F. Thus F, being closed, bounded and convex, is weakly compact.

Let us define in F the following mapping

$$g : F \to R^+ , \; (R^+ = \text{nonnegative real numbers})$$

$$g(y) = \inf_n \| T^n x_0 - y \| = \lim_{n \to \infty} \| T^n x_0 - y \| . \tag{1}$$

Since the sequence $\{\| T^n x_0 - y \|\}$ is non-increasing, therefore in (1) we have lim = inf. The mapping g so defined is continuous. Indeed,

$$g(z) = \lim \| T^n x_0 - z \| \leq \lim \| T^n x_0 - y \| + \| y - z \| = \tag{2}$$

$$= g(y) + \| y - z \| .$$

From (2) we obtain

$$| g(y) - g(z) | \leq \| y - z \| .$$

We claim that g is lower semicontinuous. In virtue of Lemma 4.2 it is enough to show that g is convex. Indeed,

$$g(ay + (1 - a) z) = \lim \| T^n x_0 - (ay + (1 - a) z) \| =$$
$$= \lim \| (1 - a) T^n x_0 + a T^n x_0 - ay - (1 - a) z \|$$
$$\leq a \cdot \lim \| T^n x_0 - y \| + (1 - a) \lim \| T^n x_0 - z \| = ag(y) + (1 - a) g(z) .$$

Thus g is lower semicontinuous. Applying Lemma 4.1 we conclude that there exists a point u in F such that

$$g(u) = d = \inf_{y \in F} g(y) .$$

Now we claim that u is unique. Suppose not, i.e. there exists an other point v in F such that $g(v) = d$. By the convexity of g it follows that $g[au + (1 - a) v] = d$ for all $0 \leq a \leq 1$. So $\| x_n - u \| \to d$, $\| x_n - v \| \to d$ and $\| x_n - (au + (1 - a) v \| \to d$. Since every Hilbert space is uniformly convex, thus by the uniform convexity of H we infer that $\| (x_n - u) - (x_n - v) \| \to 0$, i.e. $u = v$.

Finally, it remains to show that $\{T^n x_0\}$ converges weakly to u. To this effect we prove that given any subsequence of $\{T^n x_0\}$, it contains a further subsequence which converges to u. In fact given any subsequence of $\{T^n x_0\}$, it follows that it contains a further subsequence $\{T^{n(j)} x_0\}$ which converges weakly to some point p. We claim that $u = p$. Indeed, we have

$$\| T^{n(j)} x_0 - u \|^2 = \| T^{n(j)} x_0 - p \|^2 + \| p - u \|^2 + 2 \operatorname{Re} (T^{n(j)} x_0 - p , p - u).$$

Taking limit we obtain

$$g(u) = g(z) + \| p - u \|$$

which is possible only if $p = u$, whence the Theorem.

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