Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Swarupchand M. Shah

Analytic Functions with Some Derivatives Univalent and a Related Conjecture

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.5, p. 344–353.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_5_344_0>

Funzioni di variabile complessa. — *Analytic Functions with Some Derivatives Univalent and a Related Conjecture.* Nota di
SWARUPCHAND M. SHAH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dimostra che se f è una funzione analitica nel disco unitario D, se la successione $\{n_p\}$ è tale che $0 = n_0 < n_1 < \cdots < n_p < \cdots$, se ogni $f^{(n_p)}$ è univalente su D, si possono allora assegnare condizioni sufficienti perché la f sia una funzione intera di tipo esponenziale.

1. INTRODUCTION

Let f be regular in the unit disk $D = \{z \mid |z| < 1\}$ and let $\{n_p\}_{p=1}^{\infty}$ be a strictly increasing sequence of positive integers. Suppose than each $f^{(n_p)}$ is univalent in D. If

$$n_{p+1} - n_p = o(n_p), \tag{1.1}$$

Shah and Trimble [4] have shown that f must be entire (that is, f must be the restriction of an entire function to D). If

$$\limsup_{p\to\infty} (n_{p+1} - n_p) = \mu < \infty, \tag{1.2}$$

then f must be entire and of exponential type $T \left(= \limsup_{r\to\infty} \log M(r, f)/r\right)$ satisfying [5]

$$T \leq (.384)(\mu + 1)^{7/2}. \tag{1.3}$$

We improve and extend these results.

In analogy with Müntz theorem in Approximation theory, it was conjectured [6] that if $\Sigma\, 1/n_p = \infty$, then f must be entire. In Section 2 we give an example to disprove this conjecture. We assume, in what follows, that (i) f is regular in D, (ii) $\{n_p\}_{p=1}^{\infty}$ is a strictly increasing sequence of positive integers, (iii) $n_0 = 0$, and (iv) each $f^{(n_p)}$ is univalent in D. If f is entire and $T < \infty$, then f is called a function of exponential type.

THEOREM 1. *Let*

$$\eta_p = \sum_{j=1}^{p} \log\{(n_j - n_{j-1})!\} \quad , \quad \xi_p = \left(\log n_p - \frac{\eta_p}{n_p}\right), \quad p = 1, 2, \cdots, \tag{1.4}$$

$$\limsup_{p\to\infty} \frac{\eta_p}{n_p} = \eta \quad , \quad \lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{p}{n_p} = \begin{cases} \theta \\ \varphi \end{cases}. \tag{1.5}$$

(*) Nella seduta del 13 novembre 1976.

Then we have the following:

(i) *If* $\lim_{p\to\infty} \xi_p = \infty$, *then* f *is entire.*

(ii) *If* $\eta < \infty$, *then* $\theta \geq \varphi > 0$ *and* f *is an entire function of exponential type* T *satisfying*

$$(1.6) \qquad T \leq \exp\left\{\theta \log C + \eta\left(2 - \frac{\varphi}{\theta}\right) + 2\theta \log\left(1 + \frac{1}{\theta}\right)\right\},$$

where C, $1 \leq C \leq 1.0691$, *is a constant.*

(iii) *If* $\eta = \infty$ *and* $\lim_{p\to\infty} \xi_p = \infty$, *then* f *is entire but not necessarily of exponential type.*

(iv) *If* $\liminf_{p\to\infty} \xi_p < \infty$, *then* f *may not be an entire function.*

In this theorem we have not assumed that μ is finite. If we make this assumption, then we have better results than (1.3) and (1.6).

THEOREM 2. *Let* $\limsup_{p\to\infty} (n_{p+1} - n_p) = \mu$. *Then we have the following*:

(i) *If* $\mu = 1$, $T < 2\sqrt{3} < 3.4642$.

(ii) *If* $\mu < \infty$ *then* $\eta < \infty$ *and*

$$(1.7) \qquad T \leq \exp\left\{\theta \log C + \eta + 2\theta \log\left(1 + \frac{1}{\theta}\right)\right\}$$

$$(1.8) \qquad \leq C^{\theta}\{(\mu+1)!\,(\mu+1)\}^{1/\mu} < \max(6.1227, \mu).$$

(iii) *If* $n_{p+1} - n_p \to \mu$ *as* $p \to \infty$ *then*

$$(1.9) \qquad T \leq \{C(\mu+1)!\,(\mu+1)\}^{1/\mu} < \max(5.4064, \mu).$$

The proofs ot these theorems will be given in Sections 3 and 4. For a bound on C, see [2].

2. EXAMPLE 1

We construct a sequence $\{n_p\}_{p=1}^{\infty}$ and a function f such that f is regular in D, each $f^{(n_p)}$ univalent in D, $\Sigma\, 1/n_p = \infty$, and f has a singular point at 1. We follow the definitions η_p, η, etc. given in (1.4) and (1.5) and write for notational convenience $n_p = n(p)$, $a_k = a(k)$.

Let $\{M_k\}_{k=1}^{\infty}$ be any rapidly increasing sequence of positive integers such that $M_{k+1} \geq k\, 2^{M_k}$, $M_1 = 10^2$. Let

$$n(k) = k, \qquad 1 \leq k < M_1,$$
$$n(M_k + j) = 2^{M_k} + j, \qquad 0 \leq j < M_{k+1} - M_k,$$
$$n(M_{k+1}) = 2^{M_{k+1}}, \qquad k = 1, 2, \cdots$$

Then

$$\Sigma \frac{1}{n(p)} = \infty . \tag{2.1}$$

For if the series is convergent, then since $n(p)\uparrow$, we must have $p = o(n_p)$. But when $p = M_{k+1} - M_k - 1$, $(M_k + p)/n(M_k + p) \to 1$ as $k \to \infty$. Hence the series in (2.1) is divergent. Further,

$$\theta = 1 \quad , \quad \varphi = 0 . \tag{2.2}$$

Let (cf. [5, p. 399])

$$\begin{aligned} a_{k+1} &= \exp(\eta_p)/2^{p-1}(1 + n_p)! , && \text{if } k = n_p , \\ &= 0 && \text{otherwise;} \quad p = 1, 2, \cdots, \end{aligned}$$

$$f(z) = \sum_{k=1}^{\infty} a_k z^k. \tag{2.3}$$

We prove that the radius of convergence R of this series is 1. Now for $p \geq 1$,

$$\exp(\eta_p) = (n_p - n_{p-1})! \cdots (n_1 - n_0)! < (1 + n_p)!$$

and so $0 \leq a_k < 1$. Hence $R \geq 1$. Further

$$\log\{n(M_{k+1}) - n(M_{k+1} - 1)\}! = 2^{M_{k+1}} M_{k+1} \log 2 - 2^{M_{k+1}} + O(M_{k+1}^2), \tag{2.4}$$

and

$$\sum_{p=1}^{M_k} \log\{n(p) - n(p-1)\}! = O(M_k 2^{M_k}) . \tag{2.5}$$

Hence

$$\eta(M_{k+1}) = 2^{M_{k+1}} M_{k+1} \log 2 - 2^{M_{k+1}} + O(M_{k+1}^2) . \tag{2.6}$$

Consequently, for $N = n_p$, $p = M_{k+1}$,

$$\frac{\log a(N+1)}{N+1} = \frac{1}{n_p + 1} \{\eta_p - p \log 2 - (n_p + 1) \log(n_p + 1) + $$
$$+ n_p + O(\log n_p)\} = O(M_{k+1}^2)/2^{M_{k+1}} = o(1) .$$

This implies $R = 1$. Since $a_k \geq 0$, f has a singular point at 1. We now show that each $f^{(n_p)}$ is univalent in D. This would follow if we show that (cf. [5, p. 400])

$$\sum_{k=2}^{\infty} \frac{(n_p + k)! \, | a(n_p + k) |}{(k-1)!} \leq (n_p + 1)! \, | a(n_p + 1) | ,$$

that is, if

$$\frac{\exp(\eta_{p+1}-\eta_p)}{2\,(n_{p+1}-n_p)!}+\frac{\exp(\eta_{p+2}-\eta_p)}{2^2\,(n_{p+2}-n_p)!}+\cdots\leq 1\,. \tag{2.7}$$

Now

$$\exp(\eta_{p+1}-\eta_p)=(n_{p+1}-n_p)!$$

$$\exp(\eta_{p+2}-\eta_p)=(n_{p+2}-n_{p+1})!\,(n_{p+1}-n_p)!\leq(n_{p+2}-n_p)!$$

and so on. Hence (2.7) follows, and so each $f^{(n_p)}$ is univalent in D.

3. PROOF OF THEOREM I

(i) Let $f(z)=\sum_0^\infty a_k z^k$, $z\in D$. Since $f^{(n_p)}$ is univalent in D, we have for $p=1,2,\cdots$ and for $k=2,3,\cdots$ [4],

$$|a(n_p+k)|\leq\frac{Ckk!\,(n_p+1)!}{(n_p+k)!}\,|a(n_p+1)|\,. \tag{3.1}$$

After some simplification [4] we have

$$\limsup_{k\to\infty}|a_k|^{1/k}\leq K\limsup_{p\to\infty}\frac{1}{n_p}\sum_{j=1}^{p-1}(n_{j+1}-n_j+1)^{(n_{j+1}-n_j)/n_p} \tag{3.2}$$

where K is a constant. Denote the product in (3.2) by ξ_p^*. Then

$$\log\xi_p^*=\frac{1}{n_p}\sum_{j=1}^{p-1}(n_{j+1}-n_j)\{\log(n_{j+1}-n_j+1)\}\leq$$

$$\leq\frac{1}{n_p}\sum_{j=1}^{p-1}(n_{j+1}-n_j)\{\log 2+\log(n_{j+1}-n_j)\}\leq$$

$$\leq\frac{1}{n_p}\left\{(\log 2)\,n_p+\sum_{j=1}^{p-1}(n_{j+1}-n_j)\log(n_{j+1}-n_j)\right\}.$$

Now for $n\geq 1$, $n\log n\leq+n\log n!-1$. Hence

$$\sum_{j=1}^{p-1}(n_{j+1}-n_j)\log(n_{j+1}-n_j)<\sum_{j=1}^{p-1}\{(n_{j+1}-n_j)+\log((n_{j+1}-n_j)!)-1\}$$

$$=n_p-n_1+\eta_p-(p-1)\,.$$

Hence

$$0 \leq \log \xi_p^* \leq \log 2 + 1 + \frac{\eta_p}{n_p}, \tag{3.3}$$

$$\limsup_{p \to \infty} \frac{\xi_p^*}{n_p} \leq 2e \limsup_{p \to \infty} \frac{1}{n_p} \exp\left(\frac{\eta_p}{n_p}\right) =$$

$$= 2e \limsup_{p \to \infty} \exp\left(\frac{\eta_p}{n_p} - \log n_p\right).$$

Hence from (3.2), (3.3) and our hypothesis, it follows that f must be entire.

(ii) We first prove the following

LEMMA. *Let* $\{n_p\}_1^\infty$ *be any strictly increasing sequence of positive integers and let* η_p, η, θ *and* φ *be as in* (1.4) *and* (1.5). *If* $\eta < \infty$ *then* $\varphi > 0$.

Proof. Suppose if possible that $\varphi = 0$. Since $\limsup_{p \to \infty} n_p/p = \infty$, given $H > n_1$, there must be some p such that $(n_p - n_{p-1}) \geq H$. Further, there exists $\{p_k\}_1^\infty$ such that $n(p_k)/p_k > 2H$ for $k = 1, 2, \ldots$. Write $p_k = P$ and let $E = \{p \mid n_p - n_{p-1} \geq H, p \leq P\}$. Then

$$\frac{n(P)}{P} = \frac{1}{P} \sum_{p \in E} (n_p - n_{p-1}) + \frac{1}{P} \sum_{\substack{p \notin E \\ p \leq P}} (n_p - n_{p-1}) = T_1 + T_2 \quad \text{say.}$$

Since $T_2 \leq H$ and $n(P)/P > 2H$, $T_1 > H$; that is,

$$\frac{n(P)}{P} = T_1 + T_2 < 2T_1 = \frac{2}{P} \sum_{p \in E} (n_p - n_{p-1}).$$

Now for all $n \geq 1$, $\log n! \geq \frac{1}{2} n \log n$ and so

$$\frac{\eta_p}{n_p} > \frac{1}{4} \frac{\sum_{p \in E} (n_p - n_{p-1}) \log (n_p - n_{p-1})}{\sum_{p \in E} (n_p - n_{p-1})} > \frac{\log H}{4}.$$

Hence $\eta = \infty$, leading to a contradiction, and the lemma is proved.

We now prove (1.6). From (3.1) we have for $2 \leq k \leq n_{p+1} - n_p + 1$, $p = 1, 2, \cdots$ [4, p. 209],

$$|a(n_p + k)| (n_p + k)! \leq C^p D^* kk! \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)(n_{j+1} - n_j + 1)! \tag{3.4}$$

$$= C^p D^* kk! \exp(\eta_p) \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^2.$$

Here D^* is a constant. We now use the inequality for $k!$, $k \geq 2$,

$$k! \geq A \sqrt{k} \left(\frac{k}{e}\right)^k, \qquad \text{A a positive constant,}$$

and obtain

$$(3.5)\qquad t_p \equiv \frac{n_p + k}{e} \,|\, a(n_p + k) \,|^{1/(n_p+k)} \leq \left[\left(\frac{C^p D^* k}{A\sqrt{n_p + k}}\right) k! \exp(\eta_p)\right]^{1/(n_p+k)} l_p,$$

where

$$l_p = \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{2/(n_p+k)}.$$

Now

$$(3.6)\qquad (k!)^{1/(n_p+k)} < (1 + o(1)) \left(\frac{k}{e}\right)^{k/(n_p+k)} =$$

$$= (1 + o(1)) \exp\left\{\frac{k}{n_p + k} (\log k - 1)\right\} \leq$$

$$\leq (1 + o(1)) \exp\left\{\left(\frac{n_{p+1} - n_p + 1}{n_{p+1} + 1}\right) \log \frac{n_{p+1} - n_p + 1}{e}\right\},$$

since $k(\log k - 1)/(n_p + k)$ increases with k. Further,

$$(3.7)\qquad \left(\frac{n_{p+1} - n_p + 1}{e}\right)^{n_{p+1}-n_p+1} < \frac{1}{A\sqrt{n_{p+1} - n_p + 1}} (n_{p+1} - n_p + 1)! =$$

$$= \frac{\sqrt{n_{p+1} - n_p + 1}}{A} \exp(\eta_{p+1} - \eta_p).$$

Also by Jensen's inequality

$$\exp\left\{\frac{1}{p - 1} \sum_2^p \log(n_j - n_{j-1} + 1)\right\} <$$

$$< \frac{1}{p - 1} \sum_2^p (n_j - n_{j-1} + 1) = 1 + \frac{n_p - n_1}{p - 1}.$$

Consequently

$$l_p \leq \exp\left\{\frac{2(p - 1)}{n_p - n_1} \log\left(1 + \frac{n_p - n_1}{p - 1}\right)\right\}.$$

Now $(2/x)\log(1 + x) \downarrow$ in $(0, \infty)$. Hence for $p > p_0(\varepsilon)$,

$$(3.8)\qquad l_p \leq \exp\left\{2(\theta + \varepsilon) \log\left(1 + \frac{1}{\theta + \varepsilon}\right)\right\}.$$

Hence for $p > p_1$

$$t_p \leq C^{(\theta+\varepsilon)} (1 + \varepsilon) \exp\left\{\frac{\eta_{p+1} - \eta_p}{n_{p+1} + 1} + \frac{\eta_p}{n_p + k} + 2(\theta + \varepsilon) \log\left(1 + \frac{1}{\theta + \varepsilon}\right)\right\} \leq$$

$$\leq (1 + \varepsilon) \exp\left\{(\theta + \varepsilon) \log C + (\eta + \varepsilon)\left(2 - \frac{\varphi - \varepsilon}{\theta + \varepsilon}\right) + 2(\theta + \varepsilon) \log\left(1 + \frac{1}{\theta + \varepsilon}\right)\right\}$$

and so

$$(3.9) \qquad \limsup_{p\to\infty} t_p \leq \exp\left\{\theta \log C + \eta\left(2 - \frac{\varphi}{\theta}\right) + 2\,\theta \log\left(1 + \frac{1}{\theta}\right)\right\}.$$

Hence if ρ denotes the order of f, then $\rho \leq 1$. If $\rho < 1$ then $T = 0$ and (1.6) is trivially true, and if $\rho = 1$ then (3.9) gives, with the help of a known formula for the type [1, p. 11] the inequality (1.6).

(iii) From (i) it follows that f is entire. We now show that given any sequence $\{n_p\}_{p=1}^{\infty}$ satisfying the hypotheses in (iii), there exists an entire function of maximal type such that each $f^{(n_p)}$ is univalent in D.

Example 2. Let $\{n_p\}_{p=1}^{\infty}$ be a sequence satisfying $\eta = \infty = \lim\limits_{p\to\infty} \xi_p$. We follow the notations n_0, η_p, η defined in Theorem 1. Let

$$\begin{aligned} a_{k+1} &= \exp(\eta_p)/2^{p-1}(1+n_p)!\,, \qquad && k = n_p \\ &= 0\,, && \text{otherwise } p = 1, 2, \cdots, \end{aligned}$$

and let

$$(3.10) \qquad f(z) = \sum_{1}^{\infty} a_k z^k = a(n_1+1) z^{n_1+1} + a(n_2+1) z^{n_2+1} + \cdots.$$

Then

$$(3.11) \quad |a(n_p+1)|^{1/(n_p+1)} = \exp\left\{\frac{1}{n_p+1}(\eta_p - p\log 2 - n_p \log n_p + n_p) + o(1)\right\} \leq$$

$$\leq \exp\left\{\frac{1}{n_p}(\eta_p - n_p \log n_p) + O(1)\right\}$$

which tends to zero. Hence f is entire. Further,

$$\begin{aligned} \limsup_{k\to\infty} k\,|a_k|^{1/k} &= \limsup_{p\to\infty} (n_p+1)\{a(n_p+1)\}^{1/(n_p+1)} = \\ &= \limsup_{p\to\infty} \exp\left\{\log(n_p+1) + \frac{\eta_p - n_p\log n_p}{n_p} + O(1)\right\} = \\ &= \infty. \end{aligned}$$

Hence $T = \infty$. The univalence, of $f^{(n_p)}$ $(p = 1, 2, \cdots)$ in D, follows as in Example 1.

Remark. Given *any* sequence $\{n_p\}_{p=1}^{\infty}$ we can always construct an entire function f of exponential type such that each $f^{(n_p)}$ is univalent in D. We need simply take a function of the form (3.10) with coefficients $a(n_p+1) \neq 0$ and tending to zero sufficiently rapidly.

(iv) *Example 3.* We define f, as in Example 2 and assume that $\liminf\limits_{p\to\infty} \xi_p < \infty$. Then (3.11) shows that the function f defined by (3.10) is not entire. As in Example 1, each $f^{(n_p)}$ is univalent in D.

This completes the proof.

4. PROOF OF THEOREM 2

(i) The hypothesis implies that there exists an integer $N \geq 1$ such that $f^{(n)}$ is univalent for all $n \geq N$. Now the proof is similar to that of Theorem 1 of [3].

(ii) We have $n_{p+1} - n_p \leq \mu$ for $p \geq p_0$. Given $\varepsilon > 0$, there is $p_1(\varepsilon) \geq p_0$ such that (see (3.5))

$$t_p < C^{(\theta+\varepsilon)} (1 + \varepsilon) \exp (\eta_p/n_p)\, l_p \,. \tag{4.1}$$

We now use (3.8) and obtain (1.7). To prove (1.8) we require the following two inequalities.

(*a*) If $a_j \geq 0$, $t_j \geq 0$, $\Sigma t_j > 0$ and $\max\limits_{1 \leq j \leq N-1} \left(\frac{a_j}{j}\right) \leq \frac{a_N}{N}$, then

$$\frac{\sum_{j=1}^{N} a_j t_j}{\sum_{j=1}^{N} j t_j} \leq \frac{a_N}{N} \,. \tag{4.2}$$

The proof is straightforward.

(*b*) For $j = 1, 2, \cdots, \mu$; $\mu = 1, 2, \cdots$,

$$\frac{\log j! + 2 \log (j+1)}{j} \leq \frac{\log \mu! + 2 \log (\mu + 1)}{\mu} \,. \tag{4.3}$$

This is equivalent to proving that

$$(\mu + 1)^{2\mu - 2} \geq \mu!\, \mu^{\mu},$$

and the last inequality can be verified for $\mu = 1, 2, \cdots, 5$ and is easy to prove for $\mu \geq 6$. Let $p > p_0$ and $n_{j+1} - n_j = \gamma$, t_γ times for j in $[p_0, p]$, $\gamma = 1, 2, \cdots, \mu$. Then

$$\frac{\sum_{p_0}^{p-1} \{\log (n_{j+1} - n_j)! + 2 \log (n_{j+1} - n_j + 1)\}}{\sum_{p_0}^{p-1} (n_{j+1} - n_j)} = \tag{4.4}$$

$$= \frac{\sum_{\gamma=1}^{\mu} t_\gamma \{\log \gamma! + 2 \log (\gamma + 1)\}}{\sum_{\gamma=1}^{\mu} \gamma t_\gamma} \leq \frac{\log \mu! + 2 \log (\mu + 1)}{\mu}$$

by using (4.2) and (4.3). Now

$$\log \left\{ \exp \frac{\eta_p}{n_p + k} l_p \right\} \leq \frac{1}{n_p} \sum_{j=1}^{p-1} \log \{(n_{j+1} - n_j)! + 2 \log (n_{j+1} - n_j + 1)\}$$

and the first part of (1.8) follows from (3.5) and (4.4).

To prove the second part, we verify $C \{(\mu + 1)!\,(\mu + 1)\}^{1/\mu} < 6.1227$ for $\mu = 1, \cdots, 6$ and prove that it is less than μ for $\mu \geq 7$.

(iii) The proof is similar to that of part (ii) and omitted.

The proof of the theorem is complete.

5. REMARKS

(i) There are sequences $(n_p\}$ for which (1.1) does not hold but $\lim_{p \to \infty} \xi_p = \infty$ and so Theorem 1 (i) applies.

Example 4. Let $\{M_k\}$ be as in Example 1. Define $n(k) = k, 1 \leq k \leq M_1$

$$n(M_k + j) = 2n(M_k) + (j - 1), \qquad 1 \leq j \leq M_{k+1} - M_k, k = 1, 2, \ldots.$$

Then

$$\lim_{p \to \infty} \sup_{\inf} \frac{n_{p+1}}{n_p} = \begin{cases} 2 \\ 1 \end{cases}, \qquad \lim_{p \to \infty} \xi_p = \infty .$$

(ii) There are sequences $\{n_p\}$ for which (1.2) does not hold but $\eta = 0$, $\theta = \varphi = 1$ and so Theorem 1 (ii) applies.

Example 5. Let $c > 1$, $M_k = \max \{[k^c], M_{k-1} + 1\}$, $M_1 = 10^2$; $n_1 = 1$

$$\begin{aligned} n_p - n_{p-1} &= 1, && \text{if } p \neq M_k \\ &= [\log \log p], && \text{if } p = M_k, \qquad k = 1, 2, \cdots; p = 2, 3, \cdots. \end{aligned}$$

Then $\limsup_{p \to \infty} (n_p - n_{p-1}) = \infty$, $\eta = 0$, $\theta = \varphi = 1$. From (1.6) we get, for any function f regular in D for which each $f^{(n_p)}$ is univalent in D, $T \leq 4C$.

(iii) It is easy to verify that the bound for T given by (1.3) is greater than the bound for T given by Theorem 2 ((i) for the case $\mu = 1$ and (ii) for the case $\mu > 1$). Thus for $\mu = 2$, (1.3) gives $T < 17.9580$ whereas (1.8) gives $T < 4.5359$. For $\mu \geq 11$, (1.8) gives $T < (3/4\,\mu)$.

REFERENCES

[1] R. P. BOAS (1954) – *Entire Functions*, Academic Press, New York.

[2] DAVID HOROWITZ (1976) – *A refinement for coefficient estimates of univalent functions*, « Proc. Amer. Math. Soc. », *54*, 176–198.

[3] S. M. SHAH and S. Y. TRIMBLE (1969) – *Univalent functions with univalent derivatives*, « Bull. Amer. Math. Soc. », *75*, 153–157.

[4] S. M. SHAH and S. Y. TRIMBLE (1974) – *Entire functions with some derivatives univalent*, « Canad. J. Math. », *26*, 207–213.

[5] S. M. SHAH and S. Y. TRIMBLE (1974) – *The order of an entire function with some derivatives univalent*, « Jour. Math. Anal. and App. », *46*, 395–409.

[6] S. M. SHAH (1972) – *Analytic functions with univalent derivatives and entire functions of exponential type*, « Bull. Amer. Math. Soc. », *78*, 154–171.