Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Fixed point and constant mappings on metric spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.5, p. 329–332.

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<http://www.bdim.eu/item?id=RLINA_1976_8_61_5_329_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Analisi matematica. — *Fixed point and constant mappings on metric spaces.* Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se T è un'applicazione di uno spazio completo metrico X in sè tale che

$$\{d(Tx, Ty)\}^2 \leq b\,d(x, Tx)\,d(y, Ty) + c\,d(x, Ty)\,d(y, Tx), \qquad (0 \leq b, c < 1),$$

allora T ha un punto fisso e questo risulta unico.

A mapping T of a metric space X into itself is said to be a *fixed point mapping* if there exists a point z in X such that $Tz = z$ and is said to be a *constant mapping* if there exists a point z in X such that $Tx = z$ for all x in X.

The following theorem was given by Khan in [1]:

THEOREM 1. *If* T *is a mapping of the metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 \leq c\,d(x, Tx)\,d(y, Ty)$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* T *is a constant mapping.*

We first of all prove the following similar type of theorem:

THEOREM 2. *If* T *is a mapping of the metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 \leq c\,d(x, Ty)\,d(y, Tx)$$

for all x, y *in* X, *where* $0 \leq c$, *then* $T^2 = T$. *Further, if* $0 \leq c < 1$, *then* T *is a constant mapping.*

Proof. Let x be an arbitrary point in X. Then, on assuming $y = Tx$, the previous formula gives

$$\{d(Tx, T^2x)\}^2 \leq c\,d(x, T^2x)\,d(Tx, Tx) = 0 .$$

It follows that $T^2x = Tx$ for all x in X and so $T^2 = T$.

Now suppose that $c < 1$. For arbitrary points x and y in X we have

$$\{d(Tx, Ty)\}^2 = \{d(T^2x, T^2y)\}^2 \leq c\,d(Tx, T^2y)\,d(Ty, T^2x) = c\,\{d(Tx, Ty)\}^2 .$$

(*) Nella seduta del 13 novembre 1976.

Since $c < 1$, it follows that $Tx = Ty$ for all x, y in X. This means that there exists a point z in X such that $Tx = z$ for all x in X and so T is a constant mapping. This completes the proof of the Theorem.

We now prove the following theorem:

THEOREM 3. *If* T *is a mapping of the complete metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 \leq b\,d(x, Tx)\,d(y, Ty) + c\,d(x, Ty)\,d(y, Tx)$$

for all x, y *in* X, *where* $0 \leq b < 1$ *and* $0 \leq c$, *then* T *is a fixed point mapping. Further, if* $0 \leq b, c < 1$, *then the fixed point of* T *is unique.*

Proof. Let x be an arbitrary point in X. Then

$$\{d(T^n x, T^{n+1} x)\}^2 \leq b\,d(T^{n-1} x, T^n x)\,d(T^n x, T^{n+1} x) +$$

$$+ c\,d(T^{n-1} x, T^{n+1} x)\,d(T^n x, T^n x) = b\,d(T^{n-1} x, T^n x)\,d(T^n x, T^{n+1} x)$$

for $n = 1, 2, \cdots$. Thus

$$d(T^n x, T^{n+1} x) \leq b\,d(T^{n-1} x, T^n x)$$

for $n = 1, 2, \cdots$ and, since $b < 1$, it follows that $\{T^n x\}$ is a Cauchy sequence in the complete metric space X and so has a limit z.

We now have

$$\{d(T^n x, Tz)\}^2 \leq b\,d(T^{n-1} x, T^n x)\,d(z, Tz) + c\,d(T^{n-1} x, Tz)\,d(z, T^n x)$$

and, on letting n tend to infinity, we see that

$$\{d(z, Tz)\}^2 \leq 0 .$$

It follows that T is a fixed point mapping.

Now suppose that $0 \leq b, c < 1$ and that z' is a second fixed point of T. Then

$$\{d(z, z')\}^2 = \{d(Tz, Tz')\}^2 \leq b\,d(z, Tz)\,d(z', Tz') + c\,d(z, Tz')\,d(z', Tz) =$$

$$= c\,\{d(z, z')\}^2 .$$

Since $c < 1$, it follows that $z = z'$ and so the fixed point is unique. This completes the proof of the theorem.

The results of Theorem 1 and Theorem 2 suggest that the mapping. T in Theorem 3 might necessarily have to be a constant mapping if $0 \leq b + c < 1$. This however is not the case. To prove this, let $X = \{0, 1, 4\}$ and define a mapping T on X by

$$T(0) = T(1) = 0 \quad , \quad T(4) = 1 .$$

Then with metric

$$d(x, y) = |x - y|$$

for all x, y in X and with $a = b = 1/3$, it is easily proved that

$$\{d(Tx, Ty)\}^2 \leq bd(x, Tx)\, d(y, Ty) + cd(x, Ty)\, d(y, Tx)$$

for all x, y in X but T is not a constant mapping.

We now give the following theorem, the proof of which is similar to that of Theorem 3:

THEOREM 4. *If* T *is a mapping of the complete metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 \leq \max\{bd(x, Tx)\, d(y, Ty), cd(x, Ty)\, d(y, Tx)\}$$

for all x, y *in* X, *where* $0 \leq b < 1$ *and* $0 \leq c$, *then* T *is a fixed point mapping. Further, if* $0 \leq b, c < 1$, *then the fixed point of* T *is unique.*

We next prove the following theorem for compact metric spaces:

THEOREM 5. *If* T *is a continuous mapping of the compact metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 < d(x, Tx)\, d(y, Ty) + cd(x, Ty)\, d(y, Tx)$$

for all distinct x, y *in* X, *where* $0 \leq c$, *then* T *is a fixed point mapping. Further, if* $0 \leq c \leq 1$, *then the fixed point of* T *is unique.*

Proof. Since d and T are continuous functions and X is compact there exists a point z in X such that

$$d(z, Tz) = \inf\{d(x, Tx) : x \in X\}.$$

On the assumption that $Tz \neq z$, we have

$$\{d(Tz, T^2 z)\}^2 < d(z, Tz)\, d(Tz, T^2 z).$$

Now $d(Tz, T^2 z) = 0$ implies

$$d(Tz, T^2 z) < d(z, Tz)$$

contradicting the definition of z. Thus we must have $d(Tz, T^2 z) > 0$ which again implies that

$$d(Tz, T^2 z) < d(z, Tz).$$

Our assumption is therefore false and so T is a fixed point mapping.

Now suppose that $0 \leq c \leq 1$. If T has two distinct fixed points z and z', then

$$\{d(z, z')\}^2 = \{d(Tz, Tz')\}^2 < c\{d(z, z')\}^2,$$

giving a contradiction. The fixed point must therefore be unique. This completes the proof of the theorem.

We finally give the following Theorem, the proof of which is similar to that of Theorem 5:

THEOREM 6. *If* T *is a continuous mapping of the compact metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 < \max\{d(x, Tx)\, d(y, Ty),\, c d(x, Ty)\, d(y, Tx)\}$$

for all distinct x, y *in* X, *where* $0 \leq c$, *then* T *is a fixed point mapping. Further, if* $0 \leq c \leq 1$, *then the fixed point of* T *is unique.*

REFERENCE

[1] M. S. KHAN – *A theorem on fixed points*, Submitted to « Tamkand J. Math. ».