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On Ideals In (m+1)-semigroups

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Algebra. — *On Ideals In* $(m+1)$*-semigroups.* Nota di H. L. Chow, presentata (*) dal Socio G. Zappa.

Riassunto. — L'Autore, ricollegandosi alle ricerche di F. M. Sioson, studia gli ideali minimali e massimali in un $(m+1)$–semigruppo.

1. Introduction

An $(m+1)$-semigroup is an algebraic system with one $(m+1)$-ary operation from $\underbrace{S\times\cdots\times S}_{m+1}$ to S such that the associative law

$$(x_1\cdots x_{m+1})\,x_{m+2}\cdots x_{2m+1}=x_1\,(x_2\cdots x_{m+2})\cdots x_{2m+1}=\cdots$$
$$=x_1\cdots x_m\,(x_{m+1}\cdots x_{2m+1})$$

holds for $x_1,\cdots,x_{2m+1}\in S$. Trivially, S is an ordinary semigroup if $m=1$. A non-empty subset I of S is called an $(i+1)$*-ideal* if $S^i\,IS^{m-i}\subset I$, $i=0,1,\cdots,m$. By convention, $S^0\,IS^m=IS^m$ and $S^m\,IS^0=S^m\,I$. The subset I is said to be an *ideal* of S if it is an $(i+1)$-ideal for each $i=0,1,\cdots,m$. In [3], Sioson studied the ideals in $(m+1)$-semigroups and obtained various results which are extensions of those in ordinary semigroups. The present paper may be regarded as a sequel to [3].

In what follows S will denote an $(m+1)$-semigroup, and we shall investigate minimal ideals and maximal (proper) ideals in S. We introduce in § 2 then the notion of a para-ideal in S, which is a 1-ideal and $(m+1)$–ideal; we show that if S has a minimal 1-ideal or a minimal $(m+1)$-ideal, S must have a minimal para-ideal which turns out to be a minimal ideal and also a minimal $(i+1)$-ideal for $i=1,\cdots,m-1$. In § 3, maximal ideals are considered, extending some results in [1] and [2].

2. Minimal ideals

Definition. A subset I of is termed a *para-ideal* of S if I is both a 1-ideal and $(m+1)$-ideal, i.e. $IS^m\subset I$ and $S^m\,I\subset I$.

It is clear that any two para-ideals must intersect; hence S can have at most one minimal para-ideal which is clearly the intersection of all para-ideals of S.

(*) Nella seduta del 13 novembre 1976.

2.1. THEOREM. *Suppose* S *has a minimal* 1*-ideal* R.

(i) *If* R_1 *is a minimal* 1*-ideal of* S *and* $R_1 \cap R \neq \varphi$, *then* $R_1 = R$.

(ii) $R = aa_2 \cdots a_m R$ *for* $a \in R$, $a_2, \cdots, a_m \in S$.

(iii) S *has a minimal para-ideal* P *which is the union of all minimal* 1*-ideals of* S.

Proof. (i) Since $R_1 \cap R$ is a 1-ideal contained in R_1 and R, we have $R_1 = R_1 \cap R = R$.

(ii) Clearly $aa_2 \cdots a_m R \subset R$. Since $(aa_2 \cdots a_m R) S^m = aa_2 \cdots a_m (RS^m) \subset aa_2 \cdots a_m R$, i.e. $aa_2 \cdots a_m R$ is a 1-ideal of S, it follows that $aa_2 \cdots a_m R = R$.

(iii) For any para-ideal I of S, $RI^m \subset R$. Moreover, RI^m is a 1-ideal of S since $(RI^m) S^m = RI^{m-1} (IS^m) \subset RI^{m-1} I = RI^m$; hence $RI^m = R$. That $I \supset RI^m = R$ implies that the minimal para-ideal P of S exists, with $P \supset R$.

Now take $a_1 \cdots a_m \in S$; then $a_1 \cdots a_m R$ is a 1-ideal. Suppose $a_1 \cdots a_m R$ is not minimal, i.e. there is a 1-ideal R^* properly contained in $a_1 \cdots a_m R$. Let $A = R \cap \{x \in S : a_1 \cdots a_m x \in R^*\}$, and it is not difficult to check that A is a 1-ideal properly contained in R. This contradiction therefore shows that $a_1 \cdots a_m R$ is a minimal 1-ideal of S. Consequently $S^m R$ is the union of all minimal 1-ideals of S, whence $S^m R \subset P$. On the other hand, since $S^m R$ is a para-ideal of S, we get $S^m R \supset P$, and the result now follows.

It can be shown, in a similar manner, that the preceding theorem also holds, if 1-ideals are replaced by $(m+1)$-ideals.

DEFINITION. The $(m+1)$-semigroup S is called an $(m+1)$*-group* provided that, if a and any m of the symbols $x_1, \cdots, x_{m+1}$ are specified as elements of S, the equation $x_1 \cdots x_{m+1} = a$ has at least one solution in S for the remaining symbol.

2.2. THEOREM. *Let* R *be a minimal* 1*-ideal of* S *and* L *a minimal* $(m+1)$*-ideal of* S. *Then* $R \cap L$ *is an* $(m+1)$*-group.*

Proof. First we note that $R \cap L \neq \varphi$ since it contains $RS^{m-1} L$. Take $a_1, \cdots, a_m \in R \cap L$ and we see that $a_1 \cdots a_m (R \cap L) \subset R \cap L$. Suppose $a_1 \cdots a_m (R \cap L) \neq R \cap L$. Let $\mathscr{L}$ denote the set of all minimal 1-ideals of S; we then have $\cup \{R \cap L^* : L^* \in \mathscr{L}\} \neq \cup \{a_1 \cdots a_m (R \cap L^*) : L^* \in \mathscr{L}\}$, giving $R \neq a_1 \cdots a_m R$, a contradiction. Thus $R \cap L = a_1 \cdots a_m (R \cap L)$ and, similarly, $R \cap L = (R \cap L) a_1 \cdots a_m$. This together with Theorem 5.8 of [3] implies that $R \cap L$ is an $(m+1)$-group.

We may have more than one minimal 1-ideal and minimal $(m+1)$-ideal in S; but, as the next result shows, we can have at most one minimal $(i+1)$-ideal for $i = 1, \cdots, m-1$ and at most one minimal ideal in S.

2.3. LEMMA. *For each* $i = 1, \cdots, m-1$, *any two* $(i+1)$*-ideals in* S *intersect. Hence, any two ideals of* S *intersect.*

Proof. Let I, J be $(i+1)$-ideals for some $i = 1, \cdots, m-1$. Then

$$S^i (IS^{m-1} J) S^{m-i} \subset S^i (IS^m) S^{m-i} = S^i I (S^{m+1}) S^{m-i-1} \subset S^i ISS^{m-i-1} = S^i IS^{m-i} \subset I,$$

and, similarly, $S^i (IS^{m-1} J) S^{m-i} \subset J$. So $I \cap J \neq \varphi$, completing the proof.

2.4. THEOREM. *If* S *has a minimal* 1*-ideal or a minimal* $(m+1)$*-ideal, then the minimal ideal and minimal* $(i+1)$*-ideals for* $i = 1, \cdots, m-1$, *all exist and are equal to each other.*

Proof. Let I be an $(i+1)$-ideal for some $i = 1, \cdots, m-1$, i.e. $S^i IS^{m-i} \subset I$. Observe that $S^i IS^{m-i}$ is a para-ideal since

$$(S^i IS^{m-i}) S^m = S^i IS^{m-i-1} (S^{m+1}) \subset S^i IS^{m-i-1} S = S^i IS^{m-i}$$

and similarly $S^m (S^i IS^{m-i}) \subset S^i IS^{m-i}$. By virtue of Theorem 2.2 or the remark after it, S has a minimal para-ideal P. Therefore $P \subset S^i IS^{m-i} \subset I$; as a consequence, the minimal $(i+1)$-ideal K_{i+1} which is the intersection of all $(i+1)$-ideals in S must exist and $K_{i+1} \supset P$. On the other hand, since

$$P^{m+1} = P, \qquad \text{we have} \quad S^i PS^{m-i} = S^i (P^{m+1}) S^{m-i} = S^i (P^{m-i} P P^i) S^{m-i}$$
$$= (S^i P^{m-i} P) P^i S^{m-i} \subset P P^i S^{m-i} \subset P,$$

i.e. P is an $(i+1)$-ideal, whence $P \supset K_{i+1}$. Accordingly $P = K_{i+1}$.

Now let J be an ideal; then J is a para-ideal and so contains P. Hence the minimal ideal K of S exists and $K \supset P$. But P is obviously an ideal since it is an $(i+1)$-ideal for $i = 0, 1, \cdots, m$; hence $P = K$. The proof is completed.

Remark. It was shown in [3, Theorem 5.25] that, if S is a *compact* topological semigroup, then the minimal 1-ideals and minimal $(m+1)$-ideals of S must exist. In view of Theorem 2.4, we deduce that the minimal ideal and minimal $(i+1)$-ideals for $i = 1, \cdots, m-1$ also exist and are all equal. (Furthermore, they are closed). Thus, Theorem 5.25 of [3] can be improved.

DEFINITION. An $(m+1)$-semigroup S is said to be *commutative* if for any $x_1, \cdots, x_{m+1} \in S$ and each permutation f of $1, \cdots, m+1$, we have $x_1 \cdots x_{m+1} = x_{f(1)} \cdots x_{f(m+1)}$.

It is trivial that an $(i+1)$-ideal for some $i = 0, 1, \cdots, m$ is an ideal, when S is commutative.

2.5. THEOREM. *Suppose the minimal ideal* K *of* S *exists. If* S *is commutative, then* K *is an* $(m+1)$*-group.*

Proof. Take $x \in K$; then $K^m x \subset K$. It is evident that $K^m x$ is an ideal of S so that $K^m x \supset K$. Hence $K^m x = K$ and therefore $x K^m = K$. That K is an $(m+1)$-group follows from Theorem 5.8 of [3].

3. MAXIMAL IDEALS

A *maximal* ideal M of S is a proper ideal, not properly contained in any proper ideals of S; we can characterize M by considering the quotient $(m+1)$-semigroup S/M. Just like the ordinary semigroup case, the quotient S/M is defined as the $(m+1)$-semigroup which consists of the set $S \setminus M$ together with zero element o (i.e. $(S/M)^i \, o \, (S/M)^{m-i} = \{o\}$ for $i = 0, 1, \cdots, m$); see [3, p. 166].

3.1. THEOREM. *An ideal* M *of* S *is maximal if and only if* S/M *contains no proper ideals except* $\{o\}$.

Proof. The result follows from the observation that any ideal in S containing M corresponds with an ideal in S/M.

Suppose $a \in S \setminus S^{m+1}$; then $S \setminus \{a\}$ is obviously a maximal ideal of S. Following Grillet [1], we call such maximal ideals *trivial*.

3.2. THEOREM. *Let* M *be a maximal ideal of* S. *Then* M *is not trivial if and only if* M *is a prime ideal, i.e. for ideals* $I_1, \cdots, I_{m+1}$ *of* S, $I_1 \cdots I_{m+1} \subset M$ *implies* $I_j \subset M$ *for some* $j = 1, \cdots, m+1$.

Proof. We model on the proof of [2, Theorem 1] to obtain the result. First assume the maximal ideal M is prime. If M is trivial, i.e. $M = S \setminus \{a\}$ for some $a \in S \setminus S^{m+1}$, then $M \supset S^{m+1}$, implying that $M \supset S$ which is contradictory. Conversely, let M be a nontrivial maximal ideal. Let $A = S \setminus M$; then $S^{m+1} \supset A$ (for, if there exists $b \in A \setminus S^{m+1}$, then $S \setminus \{b\}$ is a maximal ideal containing M, so that $M = S \setminus \{b\}$, a contadiction). Therefore $A \subset S^{m+1} = (M \cup A)^{m+1} \subset M \cup A^{m+1}$, whence $A \subset A^{m+1}$. Now supposer M is not prime, i.e. there are ideals $I_1, \cdots, I_{m+1}$ with $I_1 \cdots I_{m+1} \subset M$ but $J_j \not\subset M$ for all $j = 1, \cdots, m+1$. It follows that $J_j \cup M = S \supset A$, giving $J_j \supset A$. So $A \subset A^{m+1} \subset I_1 \cdots I_{m+1} \subset M$, a contradiction, and the theorem is proved.

The next result is obvious.

3.3. COROLLARY. *Every maximal ideal is prime if and only if* $S = S^{m+1}$.

Assume that S has maximal ideals and denote by M^* the intersection of all maximal ideals in S. Evidently $M^* \subset S^{m+1}$. On the other hand, M^* is nonempty as the theorem below shows.

3.4. THEOREM. *Let* $\{M_\alpha : \alpha \in \Lambda\}$ *be the family of all maximal ideals in* S. *Let* $A_\alpha = S \setminus M_\alpha$ *and* $M^* = \cap M_\alpha$. *Then*

(i) $A_\alpha \cap A_\beta = \varphi$ *for* $\alpha \neq \beta$.

(ii) $S = (\cup A_\alpha) \cup M^*$.

(iii) $A_\alpha \subset M_\gamma$ *for* $\gamma \neq \alpha$.

(iv) *If* I *is an ideal of* S *and* $I \cap A_\alpha \neq \varphi$, *then* $I \supset A_\alpha$.

(v) *For* $\alpha \neq \beta$, $A_\alpha A_\beta S^{m-1} \subset M^*$.

Proof. We obtain the result by applying a similar argument to that given in [2, Theorem 2].

Finally, we examine maximal $(i+1)$-ideals for $i = 0, 1, \cdots, m$, and let M^*_{i+1} denote the intersection of all maximal $(i+1)$-ideals in S. Then (i)-(iv) of the previous theorem are still true for maximal $(i+1)$-ideals, while M^*_{i+1} may be empty. However, we shall see that $M^*_{i+1} \neq \varphi$ if $S \neq S^{m+1}$, as a direct consequence of the following theorem.

3.5. THEOREM. $S^i (S \setminus S^{m+1}) S^{m-i} \subset M^*_{i+1} \subset S^{m+1}$, $i = 0, 1, \cdots, m$.

Proof. The result is trivial if $S = S^{m+1}$. Now let $S \neq S^{m+1}$; then $M^*_{i+1} \subset S^{m+1}$, since $S \setminus \{a\}$ is a maximal $(i+1)$-ideal for each $a \in S \setminus S^{m+1}$. To prove the other inclusion, take $x_1, \cdots, x_m \in S$ and $x \in S \setminus S^{m+1}$; clearly $x_1 \cdots x_i x x_{i+1} \cdots \cdots x_m \in S \setminus \{a\}$ for any $a \in S \setminus S^{m+1}$. Now take any maximal $(i+1)$-ideal $M \neq S \setminus \{a\}$ for $a \in S \setminus S^{m+1}$. Then $x \in M$ (for, if $x \notin M$, we would have $M \subset S \setminus \{x\}$, implying that $M = S \setminus \{x\}$ which is contradictory). It follows that $x_1 \cdots x_i x x_{i+1} \cdots x_m \in S^i M S^{m-i} \subset M$. Thus $S^i (S \setminus S^{m+1}) S^{m-i} \subset M^*_{i+1}$ as required.

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