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Discussion on the existence and uniqueness of the solution of Molodensky's problem in gravity space

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Geodesia. — *Discussion on the existence and uniqueness of the solution of Molodensky's problem in gravity space.* Nota [(*)] di FERNANDO SANSÒ, presentata dal Socio L. SOLAINI.

RIASSUNTO. — In un recente lavoro l'Autore ha dato una nuova formulazione del problema di Molodensky nello spazio della gravità. Tale formulazione riduce il problema alla soluzione di una certa equazione funzionale non lineare.

In questo lavoro si applica il metodo di Newton per lo studio dell'esistenza e della unicità della soluzione di tale equazione.

INTRODUCTION

In the recent work "The boundary value problem of physical Geodesy in gravity space" the Author has proved that Molodensky's problem can be advantageously formulated in gravity space. More precisely, in that paper, the gravity potential u, for a non-rotating model of the earth, has first been considered as a function of the gravity vector $\boldsymbol{g}$. The function $u(\boldsymbol{g})$ is related to the adjoint potential $\psi(\boldsymbol{g})$ through the formula

$$g \frac{\partial \psi}{\partial g} - \psi = u(\boldsymbol{g}) \qquad \text{(Légendre transformation).}$$

It has been possible to show that the adjoint potential must satisfy a certain elliptic non linear partial differential equation, with boundary conditions of the third kind, the so called oblique derivative problem.

After some transformations this has been reduced to a Dirichlet problem for another non linear, but more manageable, partial differential equation: the new unknown in this equation is a function v whose physical meaning is that of a perturbation of the potential $u(g) = \mathrm{M}^{\frac{1}{2}} g^{\frac{1}{2}}$, that is of the potential of a homogeneous sphere of mass M.

In order to be physically acceptable, this solution v must fulfill a certain condition in the origin $\boldsymbol{g} = 0$: we will deal with this difficulty in a next note.

The purpose of this paper is to find clear and, possibly, wider conditions of existence and to study the problem of the uniqueness of the function v. This will be done by applying Newton's method to the fundamental functional equation of which v must be the solution.

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1. SUMMARY OF THE PRECEDING RESULTS

In the already mentioned work it has been proved that, under opportune conditions, the solution of Molodensky's problem is equivalent to the solution of the following fundamental functional equation.

Suppose that $u(\alpha)$ and $\boldsymbol{g}(\alpha)$ are the values of the potential and of the vector of the purely gravitational field, given on the surface of the earth: α is the two dimensional parameter $\alpha = (\Phi, \Lambda)$.

Let us put

(1,1) $\mu = M^{-\frac{1}{2}}$, M = mass of the earth (approximate value)

(1,2) $\gamma = g^{-\frac{1}{2}} \boldsymbol{g}$

(1,3) $\Gamma = \{\gamma(\alpha) = g^{-\frac{1}{2}}(\alpha)\, \boldsymbol{g}(\alpha)\}$

(1,4) D = domain (containing the origin $\gamma = 0$) with boundary Γ

(1,5) $P_\gamma = \left[\dfrac{\gamma_i \gamma_k}{\gamma^2}\right]$ (projection in the direction of γ)

(1,6) $V = [v_{ik}] = \left[\dfrac{\partial^2 v}{\partial\gamma_i \partial\gamma_k}\right] \quad ; \quad U = \left[\dfrac{\partial^2 u}{\partial\gamma_i \partial\gamma_k}\right]$

(1,7) $v(\alpha) = \dfrac{u(\alpha)}{\gamma(\alpha)} - M^{\frac{1}{2}}$

and let us also consider the bilinear operator

$$
\begin{aligned}
(1,8) \qquad B(u, v) = \Big\{ & - u\Delta v - \frac{\partial u}{\partial \gamma} \int_0^\gamma \Delta v \, d\gamma + \\
& + 2\left[Tr\left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma U d\gamma \; Tr\left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma V d\gamma - \right. \\
& \left. - Tr\left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma U d\gamma \left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma V d\gamma \right] + \\
& + 4\gamma \left[Tr\left(I - \frac{3}{4} P_\gamma\right) U Tr\left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma V d\gamma - \right. \\
& \left. - Tr\left(I - \frac{3}{4} P_\gamma\right) U \left(I - \frac{3}{4} P_\gamma\right) \int_0^\gamma V d\gamma \right] \Big\}.
\end{aligned}
$$

We will call

$$F[v] = B(v, v). \tag{1,9}$$

The Dirichlet problem for the fundamental equation can thus be written

$$\begin{cases} \Delta v = \mu F[v] & \gamma \in D \\ v = v(\alpha) & \gamma \in \Gamma. \end{cases} \tag{1,10}$$

If a solution of (1,1) exists and satisfies the condition

$$\nabla v(o) = 0, \tag{1,11}$$

we can construct the adjoint potential $\psi(\boldsymbol{g})$, through

$$\psi(\boldsymbol{g}) = -2(M^{\frac{1}{2}} + v(o)) + \int_0^{g^{\frac{1}{2}}} [v(\gamma) - v(o)]\, \gamma^{-2}\, d\gamma; \tag{1,12}$$

from (1,12) the figure of the earth can be derived by means of

$$\boldsymbol{r}(\alpha) = \nabla_g \psi|_{\boldsymbol{g}(\alpha)}. \tag{1,13}$$

The equation (1,10) has been studied in the Banach Space $C_{2+\lambda}$, where the norm is defined by

$$\begin{cases} \| v \|_{2+\lambda} = \max\limits_{\gamma \in D} | v | + \max\limits_{i, \gamma \in D} | \partial_i v | + \max\limits_{i,k,\gamma \in D} | \partial_{ik} v | + \max\limits_{i,k} H_\lambda(\partial_{ik} v) \\ H_\gamma(\partial_{ik} v) = \sup\limits_{\gamma,\gamma' D} \dfrac{| \partial_{ik} v(\gamma) - \partial_{ik} v(\gamma') |}{| \gamma - \gamma' |^\lambda} \\ \partial_i = \dfrac{\partial}{\partial \gamma_i} \quad , \quad \partial_{ik} = \dfrac{\partial^2}{\partial \gamma_i \partial \gamma_k}. \end{cases} \tag{1,14}$$

A first result on the existence of the solution of equation (1,10) has been obtained, by proving the majorization

$$\| F[v] \|_\lambda \leq h \| v \|_{2+\lambda}^2 \tag{1,15}$$

where h is a constant depending only on the shape of D and on λ, and by using Schauder's estimate

$$\begin{cases} \Delta v = f & \gamma \in D \\ v = v(\alpha) & \gamma \in \Gamma \end{cases} \Rightarrow \| v \|_{2+\lambda} \leq c\,(\| v(\alpha) \|_{2+\lambda} + \| f \|_\lambda). \tag{1,16}$$

The result is summarized in the

THEOREM I. *Let us call*

$$A(R) = \sup_{\substack{\|u\|_{2+\lambda} \leq R \\ \|v\|_{2+\lambda} \leq R}} \frac{\| F[u] - F[v] \|_\lambda}{\| u - v \|_{2+\lambda}}$$

it is easy to see that $A(R) = O(R)$.

a) *If the condition*

$$4 \mu h c^2 \| v(\alpha) \|_{2+\lambda} < 1 \tag{1,17}$$

is satisfied, then the sequence $\{v_n\}$

$$\begin{cases} \Delta v_{n+1} = \mu F[v_n] \\ v_{n+1} |_\Gamma = v(\alpha) \end{cases} \quad ; \quad \begin{cases} \Delta v_0 = 0 \\ v_0 |_\Gamma = v(\alpha) \end{cases}$$

is bounded in $C_{2+\lambda}$, *that is*

$$\| v_n \| \leq r \tag{1,16}$$

b) *If, moreover*

$$\mu A(r) c < 1 \tag{1,17}$$

then the sequence $\{v_n\}$ *is convergent in* $C_{2+\lambda}$ *to the solution* v *of* (1,10).

In order to apply Newton's method to equation (1,10) it is preferable to transform it into a fixed point equation by introducing the Green's operator G, which is the inverse of the Laplace operator with homogeneous boundary conditions.

Calling

$$Q(v) = GF[v] \tag{1,18}$$

and

$$v_0 ; \quad \Delta v_0 = 0 \quad \text{in D} \quad , \quad v_0 = v(\alpha) \quad \text{on } \Gamma \tag{1,13}$$

it is easy to prove the equivalence between (1,10) and the equation

$$v = v_0 + \mu Q(v) \tag{1,20}$$

Since from (1,16) we have

$$\| v_0 \|_{2+\lambda} \leq c \| v(\alpha) \|_{2+\lambda} \tag{1,21}$$

$$\| Q(v) \|_{2+\lambda} \leq ch \| v \|^2_{2+\lambda} \tag{1,22}$$

we recognize that (1,20) can be considered as the fixed point equation for the operator $S(v) = v_0 + \mu Q(v)$ in the Banach Space $C_{2+\lambda}$.

2. Review of Newton's method

We recall here the basic theorem [1] of Newton's method, that will be applied in the next paragraph to equation (1,20).

We consider the equation

$$x = S(x) \tag{2,1}$$

in a Banach Space E_x, under the hypothesis that $S(x)$ has a continuous Fréchet derivative in a ball $\|x - x_0\| < R$ (possibly $R = +\infty$).

Furthermore we consider the real equation

$$t = \varphi(t) \tag{2,2}$$

where $\varphi(t)$ is a continuously differentiable function in an interval (t_0, t_1), with $t_1 < t_0 + R$.

We say that equation (2,2) majorizes equation (2,1) in the interval (t_0, t_1), if

$$\begin{aligned} &\text{a)} \quad \|S(x) - x_0\| \leq \varphi(t_0) - t_0 \\ &\text{b)} \quad \|x - x_0\| \leq t - t_0 \rightarrow \|S'(x)\| \leq \varphi'(t)\,. \end{aligned} \tag{2,3}$$

We have the following

THEOREM II. *If equation* (2,2) *majorizes* (2,1) *in* (t_0, t_1) *and if* (2,2) *has at least a solution* $\bar{t}$ *in* (t_0, t_1), *then*

a) (2,1) *has at least a solution* $\bar{x}$ *in the ball*

$$\Omega = \{\|x - x_0\| \leq r = t_1 - t_0\}$$

b) *the sequence*

$$x_{n+1} = S(x_n) \tag{2,4}$$

starting with x_0 *is convergent to a solution* x^* *of* (2,1), *satisfying*

$$\|x^* - x_0\| \leq t^* - t_0 \tag{2,5}$$

where t^* *is the smallest root of* (2,2) *in* (t_0, t_1).

Moreover, *if* (2,2) *has a unique solution* t^* *in* (t_0, t_1) *and if*

$$\varphi(t_1) \leq t_1$$

then

c) (2,1) *has a unique solution* x^* *in* Ω, *satisfying* (2,5)

d) *the sequence* x_n *starting with any* $x' \in \Omega$ *is convergent to* x^*.

(1) See M. M. Vainberg, *Variational methods for the study of non linear operators*, Holden Day, 1964.

3. APPLICATION OF NEWTON'S METHOD TO THE FUNDAMENTAL EQUATION

Let us choose $E_x = C_{2+\lambda}$ and

$$S(v) = v_0 + \mu Q(v) : \tag{3,1}$$

from (1,21) and (1,22) we see that $S(v)$ is defined on $C_{2+\lambda}$. Besides we have the identity (3,2)

$$S(v + \delta v) - S(v) = \mu GB(v, \delta v) + \mu GB(\delta v, v) + \mu Q(\delta v) \tag{3,2}$$

recalling that $\| Q(\delta v) \| \leq ch \| \delta v \|^2$, from the definition of Fréchet derivative, we get

$$S'(v) = \mu Q'(v) = \mu GB(v, \cdot) + \mu GB(\cdot, v) .$$

In analogy with (1,15), it is easy to prove that

$$\| B(u, v) \|_\lambda \leq h \| u \|_{2+\lambda} \| v \|_{2+\lambda}$$

so that, recalling (1,16) we derive

$$\| S'(v) \| \leq 2 \mu ch \| v \| . \tag{3,3}$$

The inequality (3,3) proves that $S'(v)$ is defined and continuous.

Let us now consider a parameter t, such that

$$\| v - v_0 \| \leq t \qquad (t_0 = 0) \tag{3,4}$$

and the real function

$$\varphi(t) = \mu ch (\| v_0 \| + t)^2. \tag{3,5}$$

We have, for any $t \geq 0$,

$$\| S(v_0) - v_0 \| = \| \mu Q(v_0) \| \leq \mu ch \| v_0 \|^2 = \varphi(0) ,$$

$$\| v - v_0 \| \leq t \rightarrow \| S'(v) \| \leq 2 \mu ch \| v \| \leq 2 \mu ch (\| v_0 \| + t) = \varphi'(t) :$$

therefore we can conclude that:

the real equation

$$t = \varphi(t) = \mu ch (\| v_0 \| + t)^2 \tag{3,6}$$

majorizes equation (1,20) *in any interval* $(0, t_1)$, *that is any ball of* $C_{2+\lambda}$

$$\Omega = \{\| v - v_0 \| \leq t_1\} .$$

Equation (3,6), *on condition that*

$$4 \mu ch \| v_0 \| \leq 1 \tag{3,7}$$

has the real roots

$$t = \frac{1 - 2\,\mu ch \, \| v_0 \| \mp \sqrt{1 - 4\,\mu ch \, \| v_0 \|}}{2\,\mu ch} = \begin{cases} r_1 \\ r_2 \end{cases} . \tag{3,8}$$

We can observe that, by means of (1,21), (3,8) becomes

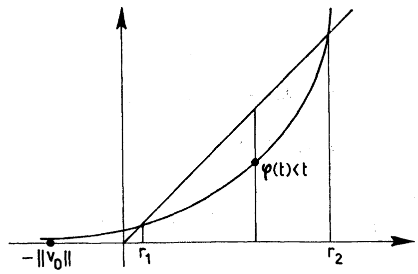

Fig. 1.

$$4\,\mu c^2 h \, \| v(\alpha) \|_{2+\lambda} \leq 1 \tag{3,9}$$

wich is almost identical (1,17).

Moreover we notice that if $(r_1 < t_1 < r_2)$ so that $\varphi(t_1) < t_1$, then there is only one root of (3,6) in $(0, t_1)$; if on the other $4\,\mu ch \, \| v_0 \| = 1$, then $r_1 = r_2 = \| v_0 \|$, so that, choosing $t_1 = \| v_0 \|$ we have $\varphi(t_1) = t_1$ and equation (3,6) has still only one solution in $(0, t_1)$. Consequently, using Theorem II of § 2, we can state the

THEOREM III. *Under the condition that*

$$4\,\mu ch \, \| v_0 \| \leq 1 \tag{3,9}$$

a) *the sequence of successive approximations starting with any* $v' \in \Omega$

$$\Omega = \{ \| v - v_0 \| < r_2 \} \tag{3,9}$$

is convergent to a solution v^* *of* (1,20):

b) v^* *is unique* Ω, *if* $r_1 < r_2$ *or in* $\overline{\Omega}$ *if* $r_1 = r_2$ *and satisfies the inequality*

$$\| v^* - v_0 \| \leq r_1 . \tag{3,10}$$

Remark. We also recall here, in short, a theorem of continuous dependence of the solution v on the boundary values $v(\alpha)$, theorem that has already been proved in "The boundary value problem of physical geodesy in gravity space".

Let us first observe that since $\| v_0 \| \leq c \, \| v(\alpha) \|_{2+\lambda}$, it is enough to prove the continuous dependence of v on v_0.

From the identity (recall also (3,2))

$$Q(v+\delta v) = Q(v) + Q'(v)\,\delta v + Q(\delta v)$$

we have

$$\delta v = \delta v_0 + \mu Q'(v)\,\delta v + \mu Q(\delta v)\,. \tag{3,11}$$

Suppose now that $4\mu ch \| v_0 \| < 1$.

Since it is $r_1 \le \| v_0 \|$, from (3,10) we get

$$\| v \| \le 2 \| v_0 \|\,.$$

Consequently, putting $P(v) = (I - \mu Q^2(v))^{-1}$

$$\| \mu Q'(v) \| \le 2\mu ch \| v \| \le 4\mu ch \| v_0 \| < 1 \to \| P(v) \| \le \frac{1}{1 - 4\mu ch \| v_0 \|}\,. \tag{3,12}$$

From (3,11) and (3,12) we derive

$$\delta v = P(v)\,\delta v_0 + \mu P(v)\,Q(\delta v)\,. \tag{3,13}$$

It is now easy to prove that if

$$\frac{4\mu ch \| \delta v_0 \|}{(1 - 4\mu ch \| v_0 \|)^2} \le 1$$

(3,13) has a solution which satisfies

$$\| \delta v \| \le \frac{2 \| \delta v_0 \|}{1 - 4\mu ch \| v_0 \|}\,. \tag{3,14}$$

Q.E.D.

4. DISCUSSION

Theorem III guarantees the uniqueness of the solution v^* only in the open ball Ω (in the closed ball $\bar{\Omega}$ if $r_2 = r_1$). Then we cannot exclude that there be other solutions $\bar{v}$, with

$$\| \bar{v} - v_0 \| \ge r_2 \qquad (> r_2 \ \text{ if } r_2 = r_1)\,. \tag{4,1}$$

Thus the problem arises if v^* is the only "a priori" correct solution from the physical point of view. We can answer in the affirmative by reasoning on the continuity of v in $v(\alpha)$.

Suppose in fact that we let $v(\alpha) \to 0$ continuously in $C_{2+\lambda}$. This corresponds to a mass configuration that tends to the spherical one, for which $u(\alpha) = M^{\frac{1}{3}} g^{\frac{1}{2}}(\alpha)$ and then $v(\alpha) = 0$. We expect that the correct solution v, which is the perturbation of the spherical potential $u = M^{\frac{1}{2}} g^{\frac{1}{2}}$, will tend continuously to zero as well. On the other hand we can observe that

$$\| v(\alpha) \|_{2+\lambda} \to 0 \;\Rightarrow\; \| v_0 \| \to 0 \;\Rightarrow\; r_1 \to 0 \,,\, r_2 \to \frac{1}{2\,\mu ch}$$

Consequently, taking (3,10) and (4,1) into account, we have

$$v^* \to 0 \text{ (continuously)}, \qquad \bar{v} \not\to 0 \left(\| \bar{v} \| \geq \frac{1}{2\,\mu ch} \right);$$

therefore we recognize v^{-*} as the unique acceptable solution.