Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Valeriu A. Sava

A regularization theorem in the theory of micropolar fluids

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.3-4, p. 242–246.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_3-4_242_0>

Meccanica dei continui. — *A regularization theorem in the theory of micropolar fluids.* Nota (*) di VALERIU A. SAVA, presentata dal Socio D. GRAFFI.

RIASSUNTO. — In base a un risultato di Agmon, Douglis, Nirenberg si stabilisce un teorema di regolarizzazione per le soluzioni deboli del problema al contorno relativo al moto stazionario di un fluido incompressibile micropolare.

In 1967 Eringen [2] proposed the theory of micropolar fluids. The vectorial forms of the field equations which govern the flow of a micropolar fluid are

$$(\lambda + \mu) \nabla\nabla\cdot v + (\mu + k) \nabla^2 v + k\nabla \times \nu - \nabla p + \rho f = \rho \dot{v}, \tag{1}$$

$$(\alpha + \beta) \nabla\nabla\cdot\nu + \gamma\nabla^2 \nu + k\nabla \times v - 2 k\nu + \rho l = \rho j\dot{\nu}, \tag{2}$$

$$\dot{\rho} + \rho\nabla\cdot v = 0, \tag{3}$$

where v is the velocity, ν the micro-rotation or spin, p the thermodynamic pressure, f and l the body-force and—couple per unit mass, ρ the density and j the micro-inertia; $\lambda, \mu, k, \alpha, \beta$, and γ are the material constants; the dot signifies material differentiation.

These equations must, of course, be supplemented by appropriate boundary and initial conditions.

Various mathematical problems for these equations have been considered by a number of Authors ([3], [4], [5]) in the past few years.

In our previous article [6] we have considered the following boundary-value problem for the differential equations of the steady flows of an incompressible micropolar fluid:

Let Ω be a lipschitzian, bounded open set in R^3 with boundary $\partial\Omega$, let $f, l \in L^2(\Omega)$ be two given vector functions. We are looking for two vector functions $u = (u_1, u_2, u_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$ and a scalar function p, representing the velocity, micro-rotation, and the pressure of the fluid, which are defined in Ω and satisfy the following equations and boundary conditions

$$(\mu + k) \nabla^2 v + k\nabla \times \nu - \nabla p + \rho f - \rho v\cdot\nabla v = 0, \quad \text{in} \quad \Omega, \tag{4}$$

$$(\alpha + \beta) \nabla\nabla\cdot\nu + \gamma\nabla^2 \nu + k\nabla\times v - 2 k\nu + \rho l - \rho j v\cdot\nabla\nu = 0, \quad \text{in} \quad \Omega, \tag{5}$$

$$\nabla\cdot v = 0, \quad \text{in} \quad \Omega, \tag{6}$$

$$v = 0 \quad , \quad \nu = 0, \quad \text{on} \quad \partial\Omega. \tag{7}$$

(*) Pervenuta all'Accademia il 6 ottobre 1976.

We have associated with (4)-(7) the variational problem:

$$
(8) \quad \begin{cases}
\text{Find } \{v , \nu\} \in \mathscr{H}_0^{1,2} \times \mathrm{H}_0^{1,2} \text{ such that} \\
(\mu + k) (\nabla v , \nabla u) - k (\nabla \times u , \nu) - \rho (v \cdot \nabla u , v) = \rho (f , u) , \\
(\alpha + \beta) (\nabla \cdot \nu , \nabla \cdot \psi) + \gamma (\nabla \nu , \nabla \psi) - k (\nabla \times v , \psi) + 2 k (\nu , \psi) - \\
\qquad - \rho j (v \cdot \nabla \psi , \nu) = \rho (l , \psi) , \\
\text{for all } \{u , \psi\} \in \mathscr{C}_0^{\infty} \times \mathrm{C}_0^{\infty}, \qquad (f , l \in \mathrm{L}^2 (\Omega)) .
\end{cases}
$$

Here C_0^{∞} is the set of vector functions defined in Ω, having continuous derivatives of all orders, and vanishing in a neighbourhood of $\partial\Omega$, $\mathscr{C}_0^{\infty}$ is the set of the vector functions in C_0^{∞} that are divergence free, $\mathrm{H}_0^{1,2}$ and $\mathscr{H}_0^{1,2}$ are the completions of C_0^{∞} and $\mathscr{C}_0^{\infty}$, respectively, in the norm

$$
{}_{\mathrm{H}}\| u \|_2 = \left(\int_{\Omega} | \nabla u |^2 \, \mathrm{d}x \right)^{1/2} .
$$

DEFINITION. *A field of pairs $(v (x) , \nu (x))$ over Ω is said to be a weak solution of boundary-value problem* (4)-(7) *whenever it is a solution of problem* (8).

The main result of [6] was the

THEOREM 1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$. Let f , l be such that $\int_{\Omega} (f \cdot u + l \cdot \psi) \, \mathrm{d}x$ be bounded in $\mathscr{H}_0^{1,2} \times \mathrm{H}_0^{1,2}$. Let $\mu , k , \alpha , \beta , \gamma$ be choosen so that*

$$
\mu > 0 \quad , \quad k > 0 \quad , \quad 3 \alpha + 2 \gamma > 0 \quad , \quad - \gamma < \beta < \gamma
$$

hold.

Then there exists a solution of problem (8) *satisfying*

$$
({}_{\mathrm{H}}\| v \|_2^2 + {}_{\mathrm{H}}\| \nu \|_2^2)^{1/2} \leq \frac{1}{c} | \{f , l\} |
$$

where $| \{f , l\} |$ is the norm of the linear continuous functional defined by $\{f , l\}$.

In this Note we shall give a regularization theorem for the weak solution of boundary-value problem (4)-(7).

We first prove

THEOREM 2. *Let Ω be an open bounded set of class c^r, $r = \max (m + 2 , 2)$, m integer, $m > 0$. Let us suppose that*

$$
v \in \mathrm{H}^{1,\alpha} (\Omega) \quad , \quad \nu \in \mathrm{H}^{1,\alpha} (\Omega) \quad , \quad p \in \mathrm{L}^{\alpha} (\Omega) \quad , \quad 1 < \alpha < + \infty ,
$$

are solutions of the problem

$$
(9)\qquad \begin{cases} (\mu + k)\,\nabla^2 v + k\nabla\times\nu - \nabla p = f\,, & \text{in } \Omega\,, \\ (\alpha+\beta)\,\nabla\nabla\cdot\nu + \gamma\nabla^2\nu + k\nabla\times v - 2\,k\nu = l\,, & \text{in } \Omega\,, \\ \nabla\cdot v = 0\,, \quad \text{in } \Omega \end{cases}
$$

$$
(10)\qquad v = v_0 \quad , \quad \nu = \nu_0\,, \quad \text{on } \partial\Omega\,.
$$

If $f\,, l \in H^{m,\alpha}(\Omega)$ *and* $v_0\,, \nu_0 \in H^{m+2-1/\alpha,\alpha}(\partial\Omega)$, *then*

$$
(11)\qquad v \in H^{m+2,\alpha}(\Omega) \quad , \quad \nu \in H^{m+2,\alpha}(\Omega) \quad , \quad p \in H^{m+1,\alpha}(\Omega)\,,
$$

and there exists a constant $c_0(\alpha\,, n\,, m\,, \Omega)$ *such that*

$$
(12)\qquad \|v\|_{H^{m+2,\alpha}(\Omega)} + \|\nu\|_{H^{m+2,\alpha}(\Omega)} + \|p\|_{H^{m+1,\alpha}(\Omega)} \le
$$

$$
\le c_0 \{\|f\|_{H^{m,\alpha}(\Omega)} + \|l\|_{H^{m,\alpha}(\Omega)} + \|v_0\|_{H^{m+2-1/\alpha,\alpha}(\partial\Omega)} +
$$

$$
+ \|\nu_0\|_{H^{m+2-1/\alpha,\alpha}(\partial\Omega)} + \|v\|_\alpha + \|\nu\|_\alpha\}\,.
$$

Proof. This theorem results from the paper of Agmon-Douglis-Nirenberg [1].

Let us denote

$$
(v_1\,, v_2\,, v_3\,, \nu_1\,, \nu_2\,, \nu_3\,, p) = (u_1\,, u_2\,, \cdots, u_7)\,,
$$

$$
F = (-f_1\,, -f_2\,, -f_3\,, -l_1\,, -l_2\,, -l_3\,, 0)\,.
$$

Then equations (9) become

$$
(13)\qquad \sum_{j=1}^{7} l_{ij}(D)\,u_j = F_i\,, \qquad 1 \le i \le 7\,,
$$

where $D = \left(\frac{\partial}{\partial x_1}\,, \frac{\partial}{\partial x_2}\,, \frac{\partial}{\partial x_3}\right)$, and $[l_{ij}(\xi)]\,, \xi = (\xi_1\,, \xi_2\,, \xi_3)$ is the matrix of elements

$$
\begin{aligned}
& l_{ij}(\xi) = (\mu + k)\,|\xi|^2\,\delta_{ij}\,, && 1 \le i\,,j \le 3 \\
& l_{i+3,j}(\xi) = l_{j,i+3}(\xi) = k\varepsilon_{jsi}\,\xi_s\,, && 1 \le i\,,j \le 3 \\
& l_{i+3,j+3}(\xi) = (\alpha+\beta)\,\xi_i\,\xi_j + \gamma\,|\xi|^2\,\delta_{ij} - 2\,k\delta_{ij}\,, && 1 \le i\,,j \le 3 \\
& l_{7i}(\xi) = -l_{i7}(\xi) = \xi_i \quad , \quad l_{7,i+1}(\xi) = l_{i+1,7}(\xi) = 0\,, && 1 \le i \le 3\,, \\
& l_{77}(\xi) = 0 \quad ; \quad |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2\,.
\end{aligned}
$$

Here ε_{ijk} are the components of the permutation tensor.

We take (see [1], p. 38) $s_i = 0$, $t_i = 2$, $1 \leq i \leq 6$, $s_7 = -1$, $t_7 = 1$. As requested, degree $l_{ij}(\xi) \leq s_i + t_j$, $1 \leq i, j \leq 7$. The matrix $[l'_{ij}(\xi)]$ will have the elements

$$l'_{ij}(\xi) = (\mu + k)\,|\xi|^2\,\delta_{ij}, \qquad 1 \leq i, j \leq 3$$

$$l'_{i+3,j}(\xi) = l'_{j,i+3}(\xi) = 0, \qquad 1 \leq i, j \leq 3$$

$$l'_{i+3,j+3}(\xi) = (\alpha + \beta)\,\xi_i\,\xi_j + \gamma\,|\xi|^2\,\delta_{ij}, \qquad 1 \leq i, j \leq 3,$$

$$l'_{7i}(\xi) = l'_{i7}(\xi) = 0, \qquad 1 \leq i \leq 3,$$

$$l'_{77}(\xi) = 0 \quad , \quad l'_{7,i+1}(\xi) = l'_{i+1,7}(\xi) = 0, \qquad 1 \leq i \leq 3.$$

We easily compute $L(\xi) = \det[l'_{ij}(\xi)] = \gamma^2(\mu + k)(\alpha + \beta)\,|\xi|^2$, so that $L(\xi) \neq 0$ for real $\xi \neq 0$, and this ensures the ellipticity of the system (Condition (1.5) of [1]). It is clear that (1.7) on p. 39 of [1] holds with $m = 6$. The Supplementary Condition on L is satisfied: $L(\xi + \tau\xi') = 0$ has exactly 6 roots with positive imaginary part and these roots are all equal to

$$\tau^+(\xi, \xi') = -\xi\cdot\xi' + i\sqrt{|\xi^2|\,|\xi'|^2 - |\xi\cdot\xi'|^2}.$$

Concerning the boundary conditions ([1], p. 42), there are 6 boundary conditions and $B_{hj} = \delta_{hj}$, $1 \leq h \leq 6$, $1 \leq j \leq 7$. We take $r_h = -2$, $h = 1, 2, \cdots, 6$. Then, as requested, degree $[B_{hj}] \leq r_h + t_j$ and we have $B'_{hj} = B_{hj}$.

It remains to check the "Complementing Boundary Condition" ([1]). It is easy to verify that

$$M^+(\xi) = (\tau - \tau^+(\xi))^6,$$

where $\tau^+(\xi) = \tau^+(\xi, \eta)$. The matrix with elements $\sum_{j=1}^{N} B_{hj}(\xi)\,L^{jk}(\xi)$ is simply the matrix with elements $l'_{hk}(\xi)$, $1 \leq h, k \leq 6$, $-l'_{h7}(\xi)$, $1 \leq h \leq 3$. A combination $\sum_{h=1}^{6} c_h \sum_{j=1}^{7} B_{hj}(\xi)\,L^{jk}(\xi)$ is then equal to an one row matrix

$$\Big[c_1(\mu + k)(\xi + \tau n)^2;\ c_2(\mu + k)(\xi + \tau n)^2;\ c_3(\mu + k)(\xi + \tau n)^2;$$

$$c_4\,\gamma(\xi + \tau n)^2 + (\xi_1 + \tau n_1)\sum_{j=1}^{3} c_{j+3}(\xi_j + \tau n_j);$$

$$c_5\,\gamma(\xi + \tau n)^2 + (\xi_2 + \tau n_2)\sum_{j=1}^{3} c_{j+3}(\xi_j + \tau n_j);$$

$$c_6\,\gamma(\xi + \tau n)^2 + (\xi_3 + \tau n_3)\sum_{j=1}^{3} c_{j+3}(\xi_j + \tau n_j);\ \sum_{j=1}^{3} c_j(\xi' + \tau n_j)\Big].$$

This matrix is zero modulo M^+ only if $c_1 = \cdots = c_6 = 0$, and the Complementing Condition holds.

We then apply Theorem 2.5, p. 78 of [1] in order to get (11) and (12).

Remark. Theorem 2 does not assert the existence of v, ν, p satisfying (9)-(12) (for given f, l, v_0, ν_0) but gives only a result on the regularity of an eventual solution.

THEOREM 3. *Let* Ω *be an open set of class* C^∞ *in* R^3 *and let* f, l *be given in* $C^\infty(\overline{\Omega})$.

Then any solution $\{v, \nu, p\}$ *of* (8) *belongs to* $\mathscr{C}^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

Proof. We notice that $\{v, \nu\} \in L^6 \times L^6$ and then

$$v \cdot \nabla v \in L^{3/2}(\Omega) \quad , \quad v \cdot \nabla \nu \in L^{3/2}(\Omega).$$

Theorem 2 implies that $\{v, \nu\} \in H^{2,3/2}(\Omega) \times H^{2,3/2}(\Omega)$, $p \in H^{1,3/2}(\Omega)$; but from the imbedding Sobolev's theorems it follows that $\{v, \nu\} \in L^\alpha(\Omega) \times L^\alpha(\Omega)$ for any α, $1 \leq \alpha < +\infty$. Therefore $D_i(v_i v_j) \in H^{-1,\alpha}(\Omega)$ and $D_i(v_i \nu_i) \in H^{-1,\alpha}(\Omega)$, for any α. The nonlinear terms $\sum_i v_i D_i v$ and $\sum_i v_i D_i \nu$ are equal to $\sum_i D_i(v_i v)$ and $\sum_i D_i(v_i \nu)$ respectively, because of (6). Hence $v \cdot \nabla v \in H^{-1,\alpha}(\Omega)$ and $v \cdot \nabla \nu \in H^{-1,\alpha}(\Omega)$ for any α. Then the Theorem 2 implies that $\{v, \nu\} \in H^{1,\alpha}(\Omega) \times H^{1,\alpha}(\Omega)$ and $p \in L^2(\Omega)$, for any α. By imbedding Sobolev's theorem $H^{1,\alpha}(\Omega) \subset L^\alpha(\Omega)$ for any $\alpha > 2$. Therefore $v \cdot \nabla v$, $v \cdot \nabla \nu \in L^\alpha(\Omega)$ for any α. Then, Theorem 2 shows us that $\{v, \nu\} \in H^{2,\alpha}(\Omega) \times H^{2,\alpha}(\Omega)$, $p \in H^{1,\alpha}(\Omega)$, for any α. Repeating this procedure we find in particular that $\{v, \nu\} \in H^{m,\alpha}(\Omega) \times H^{m,\alpha}(\Omega)$, $p \in H^{m,\alpha}(\Omega)$, for any $m \geq 1$. The same properties hold for any derivative of v, ν or p. The imbedding Sobolev's theorem implies therefore that any derivative of v, ν or p belongs to $C(\overline{\Omega})$, and this is the property announced.

Remark. It is clear that we can assume less regularity for f, l and obtain less regularity for v, ν and p.

REFERENCES

[1] S. AGMON, A. DOUGLIS and L. NIRENBERG (1964) – *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions,* II, «Comm. Pure Appl. Math.», *17*, 39–92.

[2] A. C. ERINGEN (1966) – *Theory of Micropolar Fluids,* «J. Math. Mech.», *16*, 1–18.

[3] F. FÖRSTE (1971) – *Zur Existenz statonärer Strömungen bei inkompressiblen mikropolaren Flüssigkeiten,* «Mber. Dt. Akad. Wiss.», *13*, 1–8.

[4] J. FÖRSTE (1971) – *Instationäre Strömung einer inkompressiblen mikropolaren Fluissigkeit,* «Mber. Dt. Akad. Wiss.», *13*, 741–748.

[5] H. LANGE (1976) – *Die Existenz von Lösung der Gleichungen, welche die Strömung inkompressibler mikropolarer Flussigkeiten beschreiben,* «ZAMM», *56*, 129–139.

[6] V. A. SAVA (1973) – *On the existence and uniqueness of the solution of a boundary-value problem in the theory of incompressible micropolar fluids,* «An. St. Univ. 'Al. I. Cuza,' Iasi, Sect. I., Mat.», *19*, 449–460.