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Calcolo delle probabilità. — *On the theory of weak convergence.* Nota di WILLIAM AMADIO, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — In questa Nota viene sviluppata la teoria della convergenza debole per le misure di probabilità definite su un'algebra di Boole. A tal fine viene usata una ben nota teoria della rappresentazione per una σ-algebra di Boole B per costruire una corrispondenza biunivoca tra una classe $\mathscr{F}$ di funzioni da R in B e una classe $\mathscr{F}'$ di funzioni a valori reali. Si usa poi questa corrispondenza per definire $\int_B f\,d\mu$ ove $f \in F$ e μ è una misura definita su B. Infine vengono generalizzati al caso delle misura definite su un'algebra di Boole un teorema di Alexandrov sulla convergenza debole e il teorema di Prohorov.

1. INTRODUCTION

In this paper, we develop the theory of weak convergence for probability measures defined on a Boolean algebra. The development of probability theory within the framework of a Boolean algebra has been suggested by Halmos (1944), Kolmogorov (1948) and Segal (1954), and the most extensive effort in this area has been Kappos (1969). It is our purpose here to generalize and extend the results of Sikorski (1949 b) on Boolean integration theory to the point where a definition of weak convergence can be formulated and to prove several important results on weak convergence in a Boolean algebra. In doing so, we shall also show that the expectation of a random variable, as defined in Kappos (1969), may be obtained using our integration procedure.

To be specific, Section 2 utilizes a well known representation theorem for a Boolean σ-algebra, B, to produce a one to one correspondence between a class, $\mathscr{F}$, of functions $f: R \to B$ and a class, $\mathscr{F}'$, of real valued point functions. The class $\mathscr{F}$ contains, as a subclass, the σ-homomorphisms of the Borel sets into B which were used in Sikorski (1949 b). The section concludes with an application of this correspondence to the mathematical theory of quantum mechanics.

Section 3 defines $\int_B f\,d\mu$ for $f \in \mathscr{F}$ and μ a contably additive measure defined on B. As was mentioned, this construction is a generalization of Sikorski (1949 b). At this point, we are also able to obtain the expectation of a random variable as defined in Kappos (1969).

In Section 4, we consider L, a lattice contained in the Boolean σ-algebra, B. We begin by defining L-continuity for functions belonging to $\mathscr{F}$ and

(*) Nella seduta dell'8 maggio 1976.

proving that continuity is preserved by the one to one correspondence of Section 2. We then show that several of the topological concepts which were generalized to a closure algebra in Sikorski (1949 a) are preserved by the representation of Section 2 and conclude the paper by generalizing a theorem of Alexandrov (1943) on weak convergence and Prohorov's Theorem to measures defined on a Boolean algebra.

2. The classes $\mathcal{F}$ and $\mathcal{F}'$

In Sikorski (1949 b), the author gives a definition of the Lebesgue integral in a Boolean σ-algebra, B. The two results which are necessary for this construction are the following:

I (Loomis (1947) and Sikorski (1948)). Let B a Boolean σ-algebra, then there exists a quotient σ-algebra X/I (of a point set χ) and a σ-isomorphism S such that S maps B into X/I.

II (Sikorski (1949 c)). For each σ-homomorphism of the Borel sets of the real line into B, there exists an X-measurable function F defined on χ such that $f(\mathrm{A}) = \mathrm{S}^{-1} \circ \mathrm{F}^{-1}(\mathrm{A})$ for every Borel set A.

If μ is a countably additive measure on B, then $\int_{\mathrm{B}} f \, d\mu$ is defined for every f above by $\int_{\mathrm{B}} f \, d\mu = \int_{\chi} \mathrm{F} \, d\mu \cdot \mathrm{S}^{-1}$ where F corresponds to f in II.

If we are to develop an integration theory which is broad enough to be consistent with the definition of expectation given in Kappos (1969) and to yield a theory of weak convergenge, then we must generalize the above result to the class of functions $\mathcal{F} = \{f : \mathrm{R} \rightarrow \mathrm{B} \text{ s.t. } f(\alpha) \uparrow \text{ as } \alpha \uparrow, \bigvee_{\alpha} f(\alpha) = e, \bigwedge_{\alpha} f(\alpha) = \varnothing \text{ and } \bigwedge_{\beta > \alpha} f(\beta) = f(\alpha)\}$. The class $\mathcal{F}$ clearly contains the σ-homomorphisms of the Borel sets into B as a subclass.

THEOREM 2.1. *Given* $f \in \mathcal{F}$, *define* $\mathrm{F} : \chi \rightarrow \mathrm{R}$ *by* $\mathrm{F}(x) = \inf\{\alpha \text{ s.t. } x \in \mathrm{S} \circ f(\alpha)\}$, *then* $\mathrm{F}^{-1}(-\infty, \alpha] = \mathrm{S} \circ f(\alpha)$ *for every* $\alpha \in \mathrm{R}$.

Proof. If $x \in \mathrm{S} \circ f(\alpha)$, then $\mathrm{F}(x) \leq \alpha$ by definition and we have $\mathrm{S} \circ f(\alpha) \subseteq \mathrm{F}^{-1}(-\infty, \alpha]$.

If $x \in \mathrm{F}^{-1}(-\infty, \alpha]$, then $\mathrm{F}(x) \leq \alpha$, i.e. $\inf\{\beta \text{ s.t. } x \in \mathrm{S} \circ f(\beta)\} \leq \alpha$. If $\inf\{\beta \text{ s.t. } x \in \mathrm{S} \circ f(\beta)\} < \alpha$, then $x \in \mathrm{S} \circ f(\beta)$ for some $\beta < \alpha$ which implies that $x \in \mathrm{S} \circ f(\alpha)$ by property 1 of $\mathcal{F}$. If $\inf\{\beta \text{ s.t. } x \in \mathrm{S} \circ f(\beta)\} = \alpha$, then $x \in \mathrm{S} \circ f(\beta)$ for $\beta > \alpha$. Therefore, $x \in \bigcap_{\beta > \alpha} \mathrm{S} \circ f(\beta) = \mathrm{S} \circ f(\alpha)$ by property 4 of $\mathcal{F}$. So, $\mathrm{S} \circ f(\alpha) \supseteq \mathrm{F}^{-1}(-\infty, \alpha]$. This completes the proof.

Let $\mathcal{F}' = \{\mathrm{F} : \chi \rightarrow \mathrm{R} \text{ s.t. } \mathrm{F}^{-1}(-\infty, \alpha] \in \mathrm{S}(\mathrm{B})\}$, then given $\mathrm{F} \in \mathcal{F}'$, it is easily shown that $f : \mathrm{R} \rightarrow \mathrm{B}$ defined by $f(\alpha) = \mathrm{S}^{-1} \circ \mathrm{F}^{-1}(-\infty, \alpha] \in \mathcal{F}$.

We continue by showing that the mapping defined in Theorem 2.1 is a one to one correspondence between $\mathscr{F}$ and $\mathscr{F}'$ and that it is equal to the mapping defined in the paragraph following Theorem 2.1.

LEMMA 2.1. *Suppose* $f \in \mathscr{F}$ *corresponds to* $F \in \mathscr{F}'$ *by Theorem* 2.1, *and* $F(x) = \beta$ *for some* $x \in \chi$, *then* $x \in S \circ f(\beta)$.

Proof. If $\beta = F(x) = \inf\{\alpha \text{ s.t. } x \in S \circ f(\alpha)\}$, then, by the monotonicity of $S \circ f$, we have $x \in \bigcap_{\beta < \alpha} S \circ f(\alpha) = S\left(\bigwedge_{\beta < \alpha} f(\alpha)\right)$. Therefore, $x \in S \circ f(\beta)$ by property 4 of $\mathscr{F}$.

THEOREM 2.2. *f. corresponding to* $F(x) = \inf. \{\alpha \text{ s.t. } x \in S \circ f(\alpha)\}$ *is a one to one mapping from* $\mathscr{F}$ *into* $\mathscr{F}'$.

Proof. Suppose f_1 corresponds to F_1 and f_2 corresponds to F_2 with $f_1 \neq f_2$, then for some α, $S \circ f_1(\alpha) \neq S \circ f_2(\alpha)$. So, for example, there exists $x \in X/I$ such that $x \in S \circ f_1(\alpha)$ but $x \notin S \circ f_2(\alpha)$ which implies that $F_1(x) = \inf\{\beta \text{ s.t. } x \in S \cdot f_1(\beta)\} \leq \alpha$ but, by Lemma 2.1, $F_2(x) > \alpha$. Therefore, $F_1 \neq F_2$, and the proof is complete.

Given $F \in \mathscr{F}'$, define $f(\alpha) = S^{-1} \circ F^{-1}(-\infty, \alpha]$. We know that $f \in \mathscr{F}$, and, by Theorem 2.1, f corresponds to $\hat{F}(x) = \inf\{\alpha \text{ s.t. } x \in S \circ f(\alpha)\} = \inf \{\alpha \text{ s.t. } x \in S \circ S^{-1} \circ F^{-1}(-\infty, \alpha]\} = \inf\{\alpha \text{ s.t. } x \in F^{-1}(-\infty, \alpha] = F(x)$. So, the mapping of Theorem 2.1 is onto $\mathscr{F}'$ and we have shown that it is a one to one correspondence.

Given $f \in \mathscr{F}$, define $F(x) = \inf\{\alpha \text{ s.t. } x \in S \circ f(\alpha)\}$. We know that $F \in \mathscr{F}'$ and, by the remark following Theorem 2.1, F corresponds to $\hat{f}(\alpha) = S^{-1} \circ F^{-1}(-\infty, \alpha]$. So, $S \circ f(\alpha) = F^{-1}(-\infty, \alpha]$. But by Theorem 2.1, $S \circ f(\alpha) = F^{-1}(-\infty, \alpha]$. So, $\hat{f} = f$ and the mapping of the paragraph following Theorem 2.1 is onto $\mathscr{F}$ and is, therefore, a one to one correspondence. The above remarks also make it clear that the two correspondences are the same, and so, from here on, we will write $f \leftrightarrow F$ if and only if f and F correspond under the mappings discussed above.

An important result in the mathematical theory of quantum mechanics is that to every self adjoint operator, T, on a Hilbert space there corresponds a spectral measure $A \to E(A)$ defined on the Borel sets of the real line such that $E((-\infty, \alpha]) = E(\alpha)$ where $\{E(\alpha)\}$ = the resolution of the identity associated with T. (Jauch (1968)). The usual proof of this theorem is difficult (Akhiezer and Glazman (1963)), but we shall show that the result is an easy consequence of the one to one correspondence established above.

It is clear that if we consider, for $\alpha \leq \beta$, $E(\alpha) \vee E(\beta) = E(\beta)$ and $E(\alpha) \wedge E(\beta) = E(\alpha)$, then $L = \{E(\alpha)\}$ is a distributive lattice. So, there exists a σ-isomorphism S mapping $\sigma(L)$, the Boolean σ-algebra generated by L into X/I, a quotient σ-algebra of a point set χ. $\sigma(L)$ will contain only orthogonal projections because any family of commuting projections which contains L and is closed under countable unions and intersections is a Boolean σ-algebra (Jauch (1968)) and therefore contains $\sigma(L)$.

Consider $f : \{(-\infty, \alpha]\} \to \sigma(L)$ defined by $f(-\infty, \alpha] = f(\alpha) = E(\alpha)$. Since $E(\alpha) \leq E(\beta)$ for $\alpha \leq \beta$, $f(\alpha) \uparrow$. Since there exists $M \in R$ such that $E(\alpha) = 1$ for $\alpha \geq M$, $\bigvee_\alpha f(\alpha) = e$ (the identity element of $\sigma(L)$). Since there exists $m \in R$ such that $E(\alpha) = 0$ for $\alpha \leq m$, $\bigwedge_\alpha f(\alpha) = \varnothing$. Finally, since $s - \lim_{\beta \to \alpha^+} E(\beta) = E(\alpha)$, $\bigwedge_{\beta > \alpha} f(\beta) = f(\alpha)$. So, $f \in \mathscr{F}$, and by Theorem 2.1, there exists $F : \chi \to R$ such that $E(\alpha) = f(\alpha) = S^{-1} \circ F^{-1}(-\infty, \alpha)$. We know that $F^{-1}(-\infty, \alpha] \in S(\sigma(L))$ for every α. Therefore, since X is a σ-algebra, $F^{-1}(A) \in S(\sigma(L))$ for any Borel set A, and we may consider $S^{-1} \circ F^{-1}$ as a mapping of the Borel sets of the real line into $\sigma(L)$.

Now, $S^{-1} \circ F^{-1}(-\infty, \infty) = 1$ since S is a σ-isomorphism, and if $A_i \cap A_j = \varnothing$, then $F^{-1}(A_i) \cap F(A_j) = \varnothing$

$$\Rightarrow S^{-1} \circ F^{-1}(A_j) \leq [S^{-1} \circ F^{-1}(A_i)]^c$$

$\Rightarrow S^{-1} \circ F^{-1}(A_i) \perp S^{-1} \circ F(A_j)$. So, if $A_i \cap A_j = \varnothing$ for $i \neq j$, then $S^{-1} \circ F^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} S^{-1} \circ F^{-1}(A_i) = \sum_{i=1}^{\infty} S^{-1} \circ F^{-1}(A_i)$ since $S^{-1} \circ F^{-1}(A_i) \perp S^{-1} \circ F^{-1}(A_j)$. Therefore, $S^{-1} \circ F^{-1}$ is the required spectral measure defined on all Borel sets.

3. The Integral

As is pointed out in the introduction to Sikorski (1949 b)), the difficulty in the generalization of the theory of the integral to the case of a Boolean algebra lies in the necessity of replacing the notion of real point function. The definition of random variable on a Boolean algebra given in Kappos (1969) presents a solution to this problem, and, as we shall see, the class $\mathscr{F}$ contains the inverse images of these random variables.

If $f \in \mathscr{F}$ and μ is a countably additive measure defined on B, define $\int_B f \, d\mu = \int_\chi F \, d\mu \circ S^{-1}$ where $f \leftrightarrow F$. From here on, we will drop the subscripts B and χ.

We shall now explore the consequences of our results in terms of Kappos (1969). Here the basic framework is a probability algebra, (B, p), where B is a Boolean σ-algebra and p is a strictly positive probability measure. A simple random variable, X, is one which is defined on a collection of elements $\{a_i \in B \text{ s.t. } i = 1, 2, \cdots, n\}$ with $\bigvee_{i=1}^{n} a_i = e$ and $a_i \wedge a_j = \varnothing$ for $i \neq j$. Such a collection is called an experiement. The function is then defined by $X(a_i) = \alpha_i \uparrow \in R$ and the expectation of X is defined by

$$E(X) = \sum_{i=1}^{n} \alpha_i \, p(a_i) .$$

If we define $X^{-1}(\alpha) = \bigvee_{X(a_i)\leq\alpha} a_i$, then clearly

$$X^{-1}(\alpha) = \begin{cases} e, & \text{if } \alpha_n \leq \alpha \\ \text{IV} & \\ \vdots & \\ \text{IV} & \\ a_1 \vee a_2, & \text{if } \alpha_2 \leq \alpha < \alpha_3 \\ \text{IV} & \\ a_1, & \text{if } \alpha_1 \leq \alpha < \alpha_2 \\ \text{IV} & \\ \varnothing, & \text{if } \alpha < \alpha_1. \end{cases}$$

So X^{-1} is a simple function belonging to $\mathscr{F}$ and

$$\int X^{-1}\, dp = \sum_{i=1}^{n} \alpha_i\, p\left(\bigvee_{k=1}^{i} a_k - \bigvee_{k=1}^{i-1} a_k\right) = \sum_{i=1}^{n} \alpha_i\, p(a_i) = E(X).$$

One can show that if X is a random variable defined on a countable experiement (X is called an elementary r.v.), then, if X posseses an expectation

$$E(X) \overset{\text{def}}{=} \sum_{i=1}^{\infty} \alpha_i\, p(a_i) = \int X^{-1}\, dp.$$

Finally in Kappos (1969), any r.v. X is said to possess an expectation if and only if there exists a sequence X_i of elementary random variables possessing expectation such that $X_i \xrightarrow{u} X$. E (X) is then defined to be $\lim_i E(X_i)$.

Since there is no underlying experiement to work with when one considers a non elementary random variable, X, the definition of $X^{-1}(\alpha)$ must be adjusted to

$$X^{-1}(\alpha) = \bigwedge_{n=1}^{\infty} (o - \lim_i \sup X_i^{-1}(\alpha + 1/n))$$

$$= \bigwedge_{n=1}^{\infty} (o - \lim \inf X_i^{-1}(\alpha + 1/n))$$

where X_i's belong to the sequence mentioned above. Kappos (1969) points out that $X^{-1}(\alpha)$ is monotone, increasing continuous from the right with $X^{-1}(-\infty) = \varnothing$ and $X^{-1}(+\infty) = e$. Therefore, X^{-1} belongs to $\mathscr{F}$. At this point, one can see that the generalization of Sikorski (1949 b) to the class $\mathscr{F}$ was critical.

Since X^{-1} belongs to $\mathscr{F}$, we know that there exists $F \in \mathscr{F}'$ such that $X^{-1} \leftrightarrow F$. Also, there exist $F_i \in \mathscr{F}'$ such that $F_i \leftrightarrow X_i^{-1}$. So, $X^{-1}(\alpha) =$

$= S^{-1} \circ F^{-1}(-\infty, \alpha]$ and $X_i^{-1}(\alpha) = S^{-1} \circ F_i^{-1}(-\infty, \alpha]$ which implies, by the definition of X^{-1},

$$(*) \qquad F^{-1}(-\infty, \alpha] = \bigcap_{n=1}^{\infty} \limsup_i F_i^{-1}(-\infty, \alpha + 1/n] = \bigcap_{n=1}^{\infty} \liminf F_i^{-1}(-\infty, \alpha + 1/n].$$

Suppose $F(x) = \beta$, then $x \in F^{-1}(-\infty, \beta]$. Bu (*), for every n, there exists I_n such that $x \in F_i^{-1}(-\infty, \beta + 1/n]$ for $i \geq I_n$.

Clearly, $x \notin F^{-1}(-\infty, \beta - 1/n]$ which implies by (*),

$$x \notin \bigcap_{k=1}^{\infty} \limsup_i F_i^{-1}(-\infty, \beta - 1/n + 1/k]$$

$$\Rightarrow \text{there exists a } k \text{ such that } x \notin \limsup F_i^{-1}(-\infty, \beta - 1/n + 1/k]$$

$$\Rightarrow \text{for } i \geq \text{some } I_k, x \notin F_i^{-1}(-\infty, \beta - 1/n + 1/k]$$

$$\Rightarrow F(x) > \beta - 1/n + 1/k > \beta - 1/n.$$

So for $i \geq \max(I_n, I_k)$, $\beta - 1/n \leq F_i(x) \leq \beta + 1/n$ for every n

$$\Rightarrow F_i(x) \to F(x), \qquad \text{for all } x \in \chi.$$

Now, if X is an elementary r.v. with $F \leftrightarrow X$, then it can be shown that $|X|^{-1} \leftrightarrow |F|$. So, since $X_i \xrightarrow{u} X$, $E(|X_n - X_k|) \leq \varepsilon$ for $n, k \geq N(\varepsilon)$ (Kappos (1969)) which implies $\int |F_n - F_k| \, d\, p \circ S^{-1} \leq \varepsilon$ for $n, k \geq N(\varepsilon)$. Since $F_i \to F$, we have $\int |F_i - F| \, dp \circ S^{-1} \to 0$

$$\Rightarrow \lim_i \int F_i \, dp \circ S^{-1} = \int F \, dp \circ S^{-1}$$

$$\Rightarrow \lim_i E(X_i) = \int X^{-1} \, dp$$

$$\Rightarrow E(X) = \int X^{-1} \, dp.$$

4. Weak Convergence.

We are now in a position to reap the rewards of our labors in the preceding sections. Let L be any δ-lattice contained in the Boolean σ-albegra. (A δ-lattice is one which is closed under countable meets).

Definition 4.1. $f \in \mathcal{F}$ is called L — continuous if and only if $f(\alpha) \in L$ and $\bigwedge_n f(\alpha - 1/n)^c \in L$ for every α.

Recalling that the class $\mathcal{F}$ contains the inverse images of random variables, X, defined on B, we see that the above definiton requires that $X^{-1}(-\infty, \alpha] \in L$ and $X^{-1}[\alpha, \infty) \in L$, for every α.

It can be shown that the mapping $\leftrightarrow$ preserves continuity and boundedness (Olmsted (1942) p. 171), and so, we make the following definition.

DEFINITION 4.2. If p_i, p are probability measures defined on B, then we will say $p_i \xrightarrow{w} p$ (p_i converges weakly to p) if and only if $\int f \, \mathrm{d}p_i \to \int f \, \mathrm{d}p$ for every bounded, L-continuous $f \in \mathscr{F}$.

Clearly, $\int f \, \mathrm{d}p_i \to \int f \, \mathrm{d}p$ for every bounded, L-continuous $f \in \mathscr{F}$ iff $\int \mathrm{F} \, \mathrm{d}p_i \circ \mathrm{S}^{-1} \to \int \mathrm{F} \, \mathrm{d}p \circ \mathrm{S}^{-1}$ for every bounded, S(L) continuous function belonging to $\mathscr{F}'$. And since every S(L) continuous function defined on χ belongs to $\mathscr{F}'$, we have $p_i \xrightarrow{w} p$ if and only if $p_i \circ \mathrm{S}^{-1} \xrightarrow{w} p \circ \mathrm{S}^{-1}$.

One consequence of the preceding equivalence is the following:

THEOREM 4.1. $p_i \xrightarrow{w} p$ *if and only if*

(1) $\lim_i p_i(\mathrm{B}) = p(\mathrm{B})$;

(2) $p(a^c) \leq \liminf_i p_i(a^c)$ *for every* $a \in \mathrm{L}$;

(3) $p(a) \geq \limsup_i p_i(a)$ *for every* $a \in \mathrm{L}$.

Proof. $p_i \xrightarrow{w} p$ iff $p_i \circ \mathrm{S}^{-1} \xrightarrow{w} p \circ \mathrm{S}^{-1}$.

iff (1) $p_i \circ \mathrm{S}^{-1}(\mathrm{S}(\mathrm{B})) \to p \circ \mathrm{S}^{-1}(\mathrm{S}(\mathrm{B})$;

(2) $p \circ \mathrm{S}^{-1}(\mathrm{CS}(a)) \leq \liminf_i p_i \circ \mathrm{S}^{-1}(\mathrm{CS}(a))$ for every $a \in \mathrm{L}$

(3) $p \circ \mathrm{S}^{-1}(\mathrm{S}(a)) \geq \limsup p \circ \mathrm{S}^{-1}(\mathrm{S}(a))$ for every $a \in \mathrm{L}$. Alexandrov (1943)).

iff (1) $\lim p_i(\mathrm{B}) = p(\mathrm{B})$

(2) $p(a^c) \leq \lim_i \inf p_i(a^c)$ for every $a \in \mathrm{L}$

(3) $p(a) \geq \lim_i \sup p_i(a)$ for every $a \in \mathrm{L}$.

In preparation for the statement of Prohorov's Theorem, we will endow B with certain structures familiar from topology.

DEFINITION 4.3. A lattice L is said to be normal if and only if a, $b \in \mathrm{L}$, $a \wedge b = \varnothing$, implies that there exists $c, d \in \mathrm{L}$ such that $a \leq c^c$, $b \leq d^c$ and $c^c \wedge d^c = \varnothing$.

So, suppose that L is a normal δ-lattice with identity such that there exists $a_i \in \mathrm{L}$, $i = 1, 2, \cdots$ with $a = \wedge a_{i_k}$ for every $a \in \mathrm{L}$ and some subsequence i_k, then since the basis is countable, any meet of elements belonging to L may be considered as a countable meet and therefore belongs to L since L is a δ-lattice. So, if we define $\{\mathrm{S}(a) \text{ s.t. } a \in \mathrm{L}\}$ to be the closed sets of X/I, then we have a topology on X/I.

Let S (a), S (b) be closed, then $a, b \in$ L. So, there exists $c, d \in$ L such that $a \leq c^c, b \leq d^c$ and $c^c \wedge d^c = \varnothing$ which implies that S $(a) \subseteq$ CS (c), S $(b) \subseteq$ CS (d) and CS $(c) \cap$ CS $(b) = \varnothing$. Therefore, X/I is a normal space.

If $a \in$ L, then $a = \wedge a_{i_k}$ for some subsequence i_k which implies that S $(a) = \wedge$ S (a_{i_k}). So, {S (a_i)} is a countable basis for closed sets. Therefore, X/I is second countable which implies that X|I is metrizable.

If we assume that $\bigwedge_n a_n \neq \varnothing$ for $a_n \in$ L and $a_{n+1} \leq a$, then $\bigcap_n$ S $(a_n) \neq \varnothing$ for S (a_n) closed, S $(a_{n+1}) \subseteq$ S (a_n). Therefore X/I is a complete metric space.

Now, let A be a family of measures on B.

DEFINITION 4.4. A is relatively compact if and only if there exists a subsequence p_i of A such that $p_i \xrightarrow{w} p$ (p need not be in A).

DEFINITION 4.5. $a \in$ B is said to be compact if and only if for every class of open elements $a_\alpha^c (a_\alpha \in$ L) such that $a \leq \bigvee_\alpha a_\alpha^c$, there exists a finite subclass $a_i^c, \cdots, a_n^c$ such that $a \leq \bigvee_{i=1}^{n} a_i^c$.

LEMMA 4.1. *a is compact in* B *if and only if* S (a) *is compact in* X/I.

Proof. Straightforward.

DEFINITION 4.6. A family of probability measures A is tight if and only if for every $\varepsilon > 0$, there exists compact a_ε such that $p(a_\varepsilon) > 1 - \varepsilon$ for every $p \in$ A.

THEOREM 4.2. *Let* L *be as above so that* S (B) = X/I *is a complete metric space. A family of probability measures,* A, *on* B *is relatively compact if and only if* A *is tight.*

Proof. A is relatively compact if and only if A$\circ$S^{-1} is relatively compact.

iff A$\circ$S^{-1} is tight since S (B) = X/I is a complete metric space (Varadarajan (1965)).

iff for every $\varepsilon > 0$, there exists compact S (a_ε) such that $p \circ$ S^{-1} (S $(a_\varepsilon)) > 1 - \varepsilon$ for every $p \circ$ S$^{-1} \in$ A$\circ$S^{-1}.

iff $p(a_\varepsilon) > 1 - \varepsilon$ for every $p \in$ A;

iff A is tight.

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