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Remarks on the oscillation of functional differential equations

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Equazioni funzionali. — *Remarks on the oscillation of functional differential equations* (*). Nota (**) di LU-SAN CHEN e CHEH-CHIH YEH (***), presentata dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori dànno condizioni sufficienti che assicurano il carattere oscillatorio, o la limitatezza di tutte le soluzioni dell'equazione funzionale differenziale

$$L_n x(t) + H(t; x[g_1(t)], \cdots, x[g_m(t)]) = Q(t).$$

1. INTRODUCTION

We consider the following n-th order ($n > 1$) functional differential equation

$$(*) \qquad L_n x(t) + H(t; x[g_1(t)], \cdots, x[g_m(t)]) = Q(t),$$

where the operator L_n is recursively defined by

$$L_0 x(t) = x(t) \quad , \quad L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t), \quad i = 1, 2, \cdots, n,$$

$$r_n(t) \equiv 1.$$

The conditions we always assume for

$$r_i (i = 1, 2, \cdots, n) \quad , \quad g_j (j = 1, 2, \cdots, m),$$

H and Q are as follows:

(i) $r_i(t) \in C[R_+ \equiv [0, \infty), R_+ \setminus \{0\}]$ *and* $\int^\infty r_i(t)\, dt = \infty$, $i = 1, 2, \cdots, n-1$;

(ii) $H(t; y_1, \cdots, y_m) \in C[R_+ \times R^m, R \equiv (-\infty, \infty)]$;

(iii) $g_j(t) \in C[R_+, R]$ *and* $\lim_{t\to\infty} g_j(t) = \infty$, $j = 1, 2, \cdots, m$;

(iv) $Q(t) \in C[R_+, R]$;

where $C[X, Y]$ will denote the set of all continuous functions

$$f: X \to Y \quad , \quad X \subseteq R \quad , \quad Y \subseteq R.$$

The purpose of this paper is to establish some new criteria concerning the oscillation of solutions of functional differential equations of (*). Recently, Kartsatos-Manougian [1] have discussed the special case $r_i(t) \equiv 1$

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$i = 1, 2, \cdots, n$ in (*). In what follows the term "solution" is always used only for such solutions $x(t)$ of (*) which are defined for all large t. Let **F** denote the family of all such solutions of (*). The oscillatory character is considered in the usual sense, i.e., a continuous real-valued function which is defined on an interval $[T, \infty)$, $T \geq 0$ is said *oscillatory* if it has no last zero and otherwise it is called *nonoscillatory*.

2. MAIN RESULTS

THEOREM 1. *Let the functions* r_i, H, g_j *and* Q *satisfy* (i)-(iv) *and, in addition, suppose that*

$$(1) \qquad \begin{cases} \forall y_j > 0 & (j = 1, 2, \cdots, m) \Rightarrow H(t; y_1, \cdots, y_m) \geq 0, \\ \forall y_j < 0 & (j = 1, 2, \cdots, m) \Rightarrow H(t; y_1, \cdots, y_m) \leq 0, \end{cases}$$

and

$$(2) \qquad \limsup_{t \to \infty} \{F(t, t_1, Q(s)) + kW_{n-1}(t, t_1)\} = +\infty,$$

$$(3) \qquad \liminf_{t \to \infty} \{F(t, t_1, Q(s)) + kW_{n-1}(t, t_1)\} = -\infty,$$

for each $(t_1, k) \in R_+ \times R$. *Then every solution* $x(t) \in$ **F** *is oscillatory, where*

$$F(t, t_1, Q(s)) \equiv \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \cdots \int_{t_1}^{s_{n-2}} r_{n-1}(s_{n-1}) \int_{t_1}^{s_{n-1}} Q(s)\, ds\, ds_{n-1} \cdots ds_1,$$

and

$$W_i(t, t_1) \equiv \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \cdots \int_{t_1}^{s_{i-1}} r_i(s_i)\, ds_i \cdots ds_1, \qquad i = 1, 2, \cdots, n-1.$$

Proof. Let $x(t)$ be a nonoscillatory solution of (*). Then $x(t)$ is either positive or negative for all large t. Assume that $x(t) > 0$ for $t > t_0$ (some) (a corresponding argument holds in case $x(t)$ is assume to be eventually negative). Then there exists $t_1 \geq t_0$ such that $g_j(t) \geq t_0$ for each $t \geq t_1$ and $j = 1, 2, \cdots, m$. Thus for $t \geq t_1$

$$(4) \qquad H(t; x[g_1(t)], \cdots, x[g_m(t)]) \geq 0.$$

It follows from (*) that for $t \geq t_1$

$$(5) \qquad L_n x(t) \leq Q(t).$$

Now, integrating (5) n-times from t_1 to t $(\geq t_1)$, we obtain

$$(6) \qquad x(t) \leq x(t_1) + L_1 x(t_1) W_1(t, t_1) + L_2 x(t_2) W_2(t, t_1) + \cdots \\ \cdots + L_{n-1} x(t_1) W_{n-1}(t, t_1) + F(t, t_1, Q(s)).$$

We see easily that $W_i(t, t_1) > 0$ for large t and because of (i),

(7) $$\lim_{t\to\infty} \frac{W_{i_1}(t_1, t_1)}{W_{i_2}(t, t_1)} = 0 , \qquad 1 \leq i_1 < i_2 \leq n-1 .$$

Therefore there exists $k > 0$ and $t_2 \geq t_1$ such that (6) implies

(8) $$\liminf_{t\to\infty} x(t) \leq \liminf_{t\to\infty} \{k W_{n-1}(t, t_1) + F(t, t_1, Q(s))\} = -\infty ,$$

a contradiction to the positivity of $x(t)$. This completes the proof of the theorem.

THEOREM 2. *Let the functions* r_i, H, g_j *and* Q *satisfy* (i)-(iv) *and in addition, suppose that for* $t \in R_+$

(9) $$\begin{cases} \forall y_j > 0 & (j = 1, 2, \cdots, m) \Rightarrow H(t; y_1, \cdots, y_m) \leq 0 , \\ \forall y_j < 0 & (j = 1, 2, \cdots, m) \Rightarrow H(t; y_1, \cdots, y_m) \geq 0 , \end{cases}$$

and that (2), (3) *are satisfied. Then every bounded solution* $x(t) \in \mathbf{F}$ *is oscillatory.*

Proof. If $x(t)$ is a bounded solution of (*) with $x(t) < 0$ for $t \geq t_0$, then (6) holds for $t_1 \geq t_0$ because $H(t; x[g_1(t)], \cdots, x[g_m(t)]) \geq 0$ for t large enough. Thus, $\liminf_{t\to\infty} x(t) = -\infty$, a contradiction to boundedness. A similar proof holds if $x(t)$ is assumed to be positive for large t. This completes the proof.

THEOREM 3. *Let the functions* r_i, H, g_j *and* Q *satisfy* (i)-(iv) *and in addition, if there exists a solution* $x(t) \in \mathbf{F}$ *for* $t \geq t_1$ *such that*

(10) $$\limsup_{t\to\infty} \{F(t, t_1, Q(s) - H(s; x[g_1(s)], \cdots, x[g_m(s)])) + \\ + k W_{n-1}(t, t_1)\} = +\infty ,$$

(11) $$\liminf_{t\to\infty} \{F(t, t_1, Q(s) - H(s; x[g_1(s)], \cdots, x[g_m(s)])) + \\ + k W_{n-1}(t, t_1)\} = -\infty$$

for any $k \in R$ *and* t_1 *large enough, then* $x(t)$ *is unbounded and oscillatory.*

Proof. Integrating (*) n-times we obtain

(12) $$x(t) = x(t_1) + L_1 x(t_1) W_1(t, t_1) + L_2 x(t_1) W_2(t, t_1) + \cdots \\ + L_{n-1} x(t_1) W_{n-1}(t, t_1) + F(t, t_1, Q(s) - H(s, x[g_1(s)], \cdots, x[g_m(s)])).$$

Then as in the proof of Theorem 1, it follows from (10), (11) and (12) that

(13) $$\liminf_{t\to\infty} x(t) = -\infty \quad , \quad \limsup_{t\to\infty} x(t) = +\infty ,$$

which proves our assertion.

COROLLARY 1. *Let the functions* r_i, H, g_j *and* Q *satisfy* (i)-(iv) *and* (3) *holds and in addition, if for every* $\lambda > 0$ *there exists* $\mu > 0$ *and* $T > 0$ *such that for* $t \geq T$

$$\sup_{|y| \leq \lambda} |H(t, x[g_1(t)], \cdots, x[g_m(t)])| \leq \mu, \tag{14}$$

where $y = \max\{x[g_1(t)], \cdots, x[g_m(t)]\}$, *then every solution* $x(t) \in \mathbf{F}$ *is unbounded.*

Proof. Let $x(t) \in \mathbf{F}$ be bounded. Then there exists $t_1 > 0$ and $\lambda > 0$ such that $|x[g_j(t)]| \leq \lambda$ for each $t \geq t_1$ and $j = 1, 2, \cdots, m$. Let μ and $T \geq t_1$ be as in (14). Then for every $t \geq T$

$$|H(t, x[g_1(t)], \cdots, x[g_m(t)])| \leq \mu. \tag{15}$$

Integrating (*) n-times we obtain

$$\begin{aligned} x(t) &= x(T) + L_1 x(T) W_1(t, T) + L_2 x(T) W_2(t, T) + \cdots \\ &\quad + L_{n-1} x(T) W_{n-1}(t, T) + F(t, T, Q(s) - H(s, x[g_1(s)], \cdots, x[g_m(s)])) \\ &\leq k W_{n-1}(t, T) + F(t, T, Q(s) + |H(s, x[g_1(s)], \cdots, x[g_m(s)])|) \\ &\leq (k + \mu) W_{n-1}(t, T) + F(t, T, Q(s)) \end{aligned} \tag{16}$$

for some $k \in R$. It is easy to see that $\liminf_{t \to \infty} x(t) = -\infty$, a contradiction.

COROLLARY 2. *Let the functions* r_i, H, g_j *and* Q *satisfy* (i)-(iv) *and* (3) *holds and in addition, if for every* $t \geq T \geq 0$

$$|H(t; x[g_1(t)], \cdots, x[g_m(t)])| \leq \mu \tag{17}$$

where $\mu > 0$ *is a fixed constant. Then every solution is* $x(t) \in \mathbf{F}$ *unbounded and oscillatory.*

Proof. The conclusion follows from the fact that (15) holds for any solution $x(t) \in \mathbf{F}$.

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