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A compactness method for a class of semi-linear Volterra integro-differential equations in Banach spaces

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Equazioni funzionali. — *A compactness method for a class of semi-linear Volterra integro-differential equations in Banach spaces* (*).

Nota (**) di ANDREA SCHIAFFINO presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questa Nota sono indicati teoremi di esistenza per soluzioni di una equazione integrodifferenziale di Volterra in uno spazio di Banach.

INTRODUCTION

Let B a complex Banach space and — A the infinitesimal generator of the analytical semigroup $\{e^{-tA} ; t \geq 0\}$; let D (A) denote the domain of A endowed by the graph topology. In this paper we shall study the existence of a solution to the Volterra integrodifferential equation:

$$\text{(PB 1)} \qquad u(0) = x \,, \; \frac{du}{dt} + Au + \int_0^t C(t-s)\, G(u(s))\, ds = g(t)$$

where C (*t*) is a family of bounded linear operators, G is a nonlinear operator in B and *g* (*t*) is a given continuous function.

The "mild" form of (PB 1) is the following:

$$\text{(PB 2)} \qquad u(t) = f(t) - \int_0^t ds \int_s^t e^{-(t-\tau)A}\, C(\tau - s)\, G(u(s))\, d\tau .$$

where $f(t) = e^{-tA} x + \int_0^t e^{-(t-\tau)A} g(\tau)\, d\tau$.

This kind of problems is the subject of several papers; a large bibliography on the most recent results can be found in [3] and [4].

In this paper we use a compactness method to prove local existence in the case in which G is continuous from a suitable Banach space between D (A) and B into B. In order that a solution to (PB 2) be "strong", that is be a solution to (PB 1), it suffices to add a Hölder-type hypothesis.

Finally we give some applications to a class of partial Volterra problems.

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§ 1. A PRELIMINARY RESULT

Let X be another Banach space, such that $D(A) \subset X \subset B$ in algebraic and topological sense. We denote by $L(B, X)$ and $L(B)$ the space of all bounded linear operators from B into X and B respectively. Moreover, we denote by $|\cdot|_B$, $|\cdot|_X$, $|\cdot|_{L(B,X)}$, $|\cdot|_{L(X)}$ the norms in $B, X, L(B, X)$ and $L(B)$ respectively.

In this section we shall prove the following preliminary result:

THEOREM 1. *Let us suppose*

i) *for every* $t > 0$ *the operator* e^{-tA} *is completely continuous from* B *into* X;

ii) *the map* $t \to C(t)$ *belongs to* $C^0(0, T; L(B))$;

iii) *the map* $t \to |e^{-tA}|_{L(B,X)}$ *is summable on* $]0, T]$.

Then the linear operator

$$(\mathscr{A}b)(t) = \int_0^t ds \int_s^t e^{-(t-\tau)A} C(\tau - s) b(s) d\tau$$

maps $C^0(0, T; B)$ *into* $C^0(0, T; X)$ *and is completely continuous.*

Proof. If $0 < h < T$ we may consider the operator

$$(\mathscr{A}_h b)(t) = \begin{cases} 0 & 0 \le t \le h \\ \displaystyle\int_0^{t-h} ds \int_s^{t-h} e^{-(t-\tau)A} C(\tau - s) b(s) d\tau & h \le t \le T. \end{cases}$$

Set $M = \max\{|C(t)|_{L(B)} ; 0 \le t \le T\}$ and denote by S_T the unit ball of $C^0(0, T; B)$; we will prove that $\mathscr{A}_h$ maps $C^0(0, T; B)$ into $C^0(0, T; X)$ compactly by means of Ascoli's theorem; first we shall prove that $\mathscr{A}_h S_T$ is an equicontinuous subset of $C^0(0, T; X)$.

Given $\varepsilon > 0$ choose $\sigma > 0$ such that

$$(1) \quad \begin{cases} h \le t_1 < t_2 \le T, & t_2 - t_1 < \sigma \Rightarrow |e^{-t_2 A} - e^{-t_1 A}|_{L(B,X)} < \varepsilon \\ 0 \le t_1 < t_2 \le T, & t_2 - t_1 < \sigma \Rightarrow \displaystyle\int_{t_1}^{t_2} |e^{-tA}|_{L(B,X)} dt < \varepsilon . \end{cases}$$

Let us consider t_1 and t_2 such that $0 \le t_1 < t_2 \le T$, $t_2 - t_1 < \sigma$ and $b(t) \in S_T$.

If $t_2 \le h$ we have $(\mathscr{A}_h b)(t_2) - (\mathscr{A}_h b)(t_1) = 0$.

If $t_1 \leq h < t_2$ we have $(\mathcal{A}_h b)(t_2) - (\mathcal{A}_h b)(t_1) = (\mathcal{A}_h b)(t_2)$; therefore:

$$|(\mathcal{A}_h b)(t_2) - (\mathcal{A}_h b)(t_1)|_X \leq M \int_0^{t_2-h} ds \int_s^{t_2-h} |e^{-(t_2-\tau)A}|_{L(B,X)} d\tau =$$

$$= M \int_0^{t_2-h} ds \int_h^{t_2-s} |e^{-\tau A}|_{L(B,X)} d\tau \leq TM \int_{t_1}^{t_2} |e^{-\tau A}|_{L(B,X)} d\tau < MT\varepsilon .$$

If $h < t_1$ we have:

$$(\mathcal{A}_h b)(t_2) - (\mathcal{A}_h b)(t_1) = \int_0^{t_1-h} ds \int_s^{t_1-h} [e^{-(t_2-\tau)A} - e^{-(t_1-\tau)A}] C(\tau - s) b(s) ds +$$

$$+ \int_0^{t_1-h} ds \int_{t_1-h}^{t_2-h} e^{-(t_2-\tau)A} C(\tau - s) b(s) ds + \int_{t_1-h}^{t_2-h} e^{-(t_2-\tau)A} C(\tau - s) b(s) ds;$$

therefore

$$|(\mathcal{A}_h b)(t_2) - (\mathcal{A}_h b)(t_1)|_X \leq \frac{MT^2 \varepsilon}{2} + M(T + \sigma) \int_h^{h+\sigma} |e^{-\tau A}|_{L(B,X)} dt \leq$$

$$\leq \left[\frac{MT^2}{2} + M(T + \sigma)\right] \varepsilon ;$$

it follows that $\mathcal{A}_h S$ is equicontinuous.

To construct a compact subset D_h of X such that $(\mathcal{A}_h b)(t) \in D_h$ when $b \in S_T$ and $0 \leq t \leq T$ let us first prove that the set

$$\Gamma_h = \bigcup_{h \leq t \leq T} e^{-tA} S_B$$

(here S_B is the unit ball in B) is a precompact subset of X.

In fact let $\{x_n\} \subset \Gamma_h$; we have $x_n = e^{-t_n A} b_n$ ($h \leq t_n \leq T$ and $b_n \in S_B$) and we may suppose $t_n \to t \geq h$ and $e^{-tA} b_n \to x$; therefore $x_n \to x$.

Let K_h be the closed convex hull of Γ_h; let us observe that K_h is balanced.

Finally we can choose $D_h = \dfrac{MT^2}{2} K_h$; in fact $(\mathcal{A}_h b)(t) = 0 \in D_h$ if $t \leq h$; in the case $t > h$ we have $C(\tau - s) b(s) \in MS_B$ and $e^{-(t-s)A} C(\tau - s) b(s) \in M\Gamma_h \subset MD_h$; since D_h is convex and balanced and

$$\int_0^{t-h} ds \int_0^{t-h} d\tau \leq \frac{T^2}{2}$$

we can apply the mean value theorem to get that $\mathcal{A}_h S_B \subset D_h$.

Theorem 1 will follow if we shall prove that

$$\lim_{t\to 0+} |(\mathcal{A}b)(t) - (\mathcal{A}_h b)(t)|_X = 0 \tag{2}$$

uniformly for $b \in S_T$ and $0 \le t \le T$.

To prove (2) let us first consider the case $t \le h$; then we have

$$|(\mathcal{A}b)(t) - (\mathcal{A}_h b)(t)|_X = |(\mathcal{A}b)(t)|_X \le Mh \int_0^h |e^{-\tau A}|_{L(B,X)}\, d\tau .$$

In the case $t > h$ we have

$$(\mathcal{A}b)(t) - (\mathcal{A}_h b)(t) = \int_0^{t-h} ds \int_{t-h}^{t} e^{-(t-\tau)A} C(\tau - s)\, b(s)\, d\tau +$$

$$+ \int_{t-h}^{t} ds \int_s^t e^{-(t-\tau)A} C(\tau - s)\, b(s)\, ds ;$$

therefore

$$|(\mathcal{A}b)(t) - (\mathcal{A}_h b)(t)|_X \le MT \int_0^h |e^{-\tau A}|_{L(B,X)}\, d\tau + Mh \int_0^h |e^{-\tau A}|_{L(B,X)}\, d\tau$$

and (2) follows. Theorem 1 is now proved.

§ 2. THE LOCAL EXISTENCE THEOREM

The Volterra equation (PB 2) can be written in the form

$$u(t) = (\mathcal{A}G)(u(t)) + f(t) \tag{PB 2'}$$

so that we may study (PB 2) by means of Schauder's fixed point theorem. More precisely, we have:

THEOREM 2. *Let us suppose, in addition to the hypotheses of Theorem 1:*

j) $G \in C^0(E, B)$ *where* E *is the ball* $\{x \in X : |x - x_0|_X \le r\}$;

jj) $g \in L^1(0, T; X)$.

Then there exists a solution to (PB 2) *in* $[0, T_0]$ *with a suitable* $T_0 \le T$.

Proof. We may suppose $|G(x)|_B \le N$, $x \in E$, because of the continuity of G. For every $T_0 \le T$ the map

$$(\mathcal{B}u)(t) = f(t) + (\mathcal{A}G)(u(t))$$

is completely continuous from $C^0(0, T_0; E)$ into $C^0(0, T_0; X)$; to apply Schauder's theorem we have to choose T_0 in such a way to get

$$\mathcal{B}(C^0(0, T_0; E)) \subset C^0(0, T_0; E) .$$

We have

$$|(\mathcal{B}u)(t) - x_0|_X \leq |e^{-tA} x_0 - x_0|_X + \int_0^t |f(\tau)|_X \, d\tau +$$

$$+ \text{MNT} \int_0^t |e^{-\tau A}|_{L(B,X)} \, d\tau$$

which is less than r if $t \leq T_0$, where $T_0 > 0$ is close to 0; the theorem follows.

Remark. If $g(t)$ belongs to $C^0(0, T; B)$, every solution to (PB 2) is a mild solution to

$$\frac{du}{dt} + Au = h(t) \tag{3}$$

where $h(t) = g(t) - \int_0^t C(t - s) G(u(s)) \, ds$ is continuous.

Therefore $u(t)$ is Hölder-continuous in the sense of the B-norm (see [6]). We can now state the following

COROLLARY. *If, in addition to the hypotheses of Theorem* 2, *we suppose:*

j)′ *g is Hölder-continuous from* $[0, T]$ *into* B;

jj)′ $C(t)$ *is Hölder-continuous from* $[0, T]$ *into* $L(B)$;

jjj)′ $x_0 \in D(A)$

we conclude that u is a strict solution to (PB 1) *and* $\frac{du}{dt}$, $(Au)(t) \in C^\alpha(0, T; B)$ *where α is the Hölder coefficient of g and* C.

Proof. It suffices to prove that the function $h(t)$ is Hölder-continuous (see [6]). Set $b(t) = G(u(t))$; we have, for $0 \leq t_1 < t_2 \leq T_0$:

$$|h(t_2) - h(t_1)|_B \leq |g(t_2) - g(t_1)|_B + N \int_{t_1}^{t_2} |C(t_2 - s)|_{L(B)} \, ds +$$

$$+ N \int_0^{t_1} |C(t_2 - s) - C(t_1 - s)|_{L(B)} \, ds \leq K_\alpha (t_2 - t_1)^\alpha$$

where K_α is a suitable constant.

§ 3. SOME APPLICATIONS

Let Ω be a bounded open set of R^n whose boundary $\partial\Omega$ is smooth. Let us consider the problem

$$(4)\quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_x u(x,t) + \displaystyle\int^t c(x,t-s)\, G(x,u(x,t-s), \\ \qquad\qquad \nabla(u(x,t-s))\, ds = g(x,t)\ x \in \Omega,\ t > 0 \\ u(x,t) = 0 \qquad x \in \partial\Omega \qquad t \geq 0 \\ u(x,0) = u_0(x) \in L^p(\Omega)\ x \in \Omega. \end{cases}$$

Let us consider $B = L^p(\Omega)$ where $p > n$, $D(A) = H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ and $A = -\Delta$; let us choose $\theta < 1$ in such a way that $D(A^\theta) \subset C^1(\overline{\Omega})$ and set $X = D(A^\theta)$. Moreover, define $(C(t)u)(x) = c(x,t)u(x)$. So by theorem 2 it follows

THEOREM 3. *Let $c(x,t)$ and $G(x,u,v)$ be continuous, respectively, in $\overline{\Omega}\cdot[0,T]$ and $\overline{\Omega}\cdot R\cdot R^n$; suppose in addition that $g(\cdot,t)$ and $\dfrac{\partial g}{\partial x_i}$ belong to $L^1(0,T;L^p(\Omega))$. Then there exists a mild local solution to problem (4).*

If moreover $c(x,t)$ and $g(x,t)$ are Hölder-continuous in t and $u_0(x)$ belongs to $D(A)$, every mild solution to problem (4) is a strict solution and $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ belong to $C^0(0,T_0;L^p(\Omega))$.

Remark 1. If $p < n$ Theorm 3 holds with the further hypothesis

$$|G(x,u,v)| \leq \gamma_0 + \gamma_1 |u|^{p_1} + \gamma_2 |v|^{p_2} \qquad x \in \overline{\Omega}\ u \in R\ v \in R^n$$

where p_1 and p_2 are suitable Sobolev exponents.

Remark 2. Let $B^0 = \{u \in C^0(\overline{\Omega}) : u(x) = 0\ x \in \partial\Omega\}$. If $u_0 \in B^0$ we can set our problem in B^0 and pose $X = |u \in C^{1+\theta}(\overline{\Omega}) : u(x) = 0\, x \in \partial\Omega|$, where $0 < \theta < 1$; we conclude that every mild solution to problem (4) belongs to $C^\alpha([0,T_0]\times\overline{\Omega})$.

Moreover, if $c(x,t)$ and $g(x,t)$ are Hölder-continuous and if $\Delta u_0 \in B^0$, every mild solution to problem (4) is strict and all its derivatives of first order are Hölder-continuous as its space-derivatives of second order.

REFERENCES

[1] V. BARBU (1973) – *Integrodifferential equations in Hilbert spaces*, «Anal. Stiin. ale Univ. 'Al. I. Cuza,' Din. Iaşi», *19*.

[2] V. BARBU (1976) – *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing and Editura Academiei, Leyden–Bucaresti.

[3] V. BARBU – *Nonlinear Volterra Integrodifferential Equations in Hilbert Space* (to appear on « Sem. Matem. Univ. », Bari).

[4] M. G. CRANDALL, S. O. LONDEN and J. A. NOHEL (1975) – *An abstract nonlinear Volterra integrodifferential equation*, « M.R.C. Tech. Summary Report, Univ. of Wisconsin », December.

[5] C. M. DAFERMOS (1970) – *An abstract Volterra equation with applications to linear viscoelasticity*, « J. Differential Equations », 7, 554–569.

[6] G. DA PRATO and P. GRISVARD (1975) – *Sommes d'operateurs lineaires et equations differentielles operationelles*, « J. Math. pures et appl. », 54.

[7] R. C. MACCAMY – *Stability theorems for a class of functional differential equations*, « S.I.A.M., J. Math. Anal. » (to appear).

[8] J. S. WONG – *A nonlinear integro-partial differential equation arising from viscoelasticity* (to appear).