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Note on distance between zeros of the n-th order nonlinear differential equations

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Equazioni differenziali ordinarie. — *Note on distance between zeros of the n-th order nonlinear differential equations* (*). Nota (**) di LU-SAN CHEN e CHEH-CHIH YEH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori estendono in questa Nota un risultato di W. T. Patula relativo all'equazione differenziale ordinaria, non lineare

$$L_n x(t) + \sum_{i=1}^{m} p_i(t)\, x(t)\, f_i(x(t)) = q(t)$$

dove l'operatore L_j è definito della formula ricorrente

$$L_0 x(t) = x(t), \qquad L_j x(t) = \frac{1}{r_j(t)} \frac{d}{dt} L_{j-1} x(t), \qquad j = 1, \cdots, n,$$
$$r_n(t) = 1.$$

We consider the following n-th order functional equation

$$L_n x(t) + \sum_{i=1}^{m} p_i(t)\, x(t)\, f_i(x(t)) = q(t) \tag{1}$$

where the operators L_j are recursively defined by

$$L_0 x = x, \qquad L_j x = \frac{1}{r_j(t)} \frac{d}{dt} L_{j-1} x, \qquad j = 1, 2, \cdots, n,$$
$$r_n(t) = 1.$$

It is assumed throughout this paper that

(i) $r_j(t) \in C[R_+, R_+ \setminus \{0\}]$, $\quad j = 1, 2, \cdots, n$,

(ii) $p_i(t), q(t) \in C[R_+, R]$,

$p_i^+(t) = \max(p_i(t), 0) \geq 0, \quad p_i^+(t) \not\equiv 0, \quad i = 1, 2, \cdots, m.$

(iii) $f_i(y) \in C[R, R]$, *for* $y > 0$, $f_i(y) = f_i(-y) > 0$, $i = 1, 2, \cdots, m$.

THEOREM 1. *Let* $\alpha_1 > \alpha_2 > \cdots > \alpha_{n-1}$ *be respectively the zeros of*

$$L_1 x(t), L_2 x(t), \ldots, L_{n-1} x(t)$$

where $x(t)$ *is a solution of* (1). *Suppose that* $b < \alpha_{n-1}$ *and* $a > \alpha_1$ *are zeros of* $x(t)$. *If* $x(t)$ *is nonnegative or nonpositive*

$$M = \max |x(t)| = |x(t_0)|, \qquad t_0 \in (b, a), \qquad K_i = \max_{y \in [-M, M]} f_i(y), \tag{2}$$
$$i = 1, 2, \cdots, m,$$

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for $t \in (b, a)$, *then*

$$(3)\qquad 1 < \int_b^{t_0} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, K_i - \frac{q(s)}{M} \right\} ds\, ds_{n-1} \cdots ds_1,$$

$$(4)\qquad 1 < \int_{t_0}^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, K_i - \frac{q(s)}{M} \right\} ds\, ds_{n-1} \cdots ds_1,$$

$$(5)\qquad 2 < \int_b^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, K_i - \frac{q(s)}{M} \right\} ds\, ds_{n-1} \cdots ds_1 .$$

Proof. On repeated integration from equation (1), we get

$$\frac{x'(t)}{r_1(t)} = L_1 x(t) - L_1 x(\alpha_1) = \int_{\alpha_1}^{t} r_2(s_2) \int_{\alpha_2}^{s_2} r_3(s_3) \cdots \int_{\alpha_{n-2}}^{s_{n-2}} \times$$
$$\times \left\{ \sum_{i=1}^m [p_i^-(s) - p_i^+(s)]\, x(s) f_i(x(s)) + q(s) \right\} ds\, ds_{n-1} \cdots ds_2 .$$

Integrating it from t_0 to t we obtain

$$(6)\qquad x(t) - x(t_0) = \int_{t_0}^{t} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \times$$
$$\times \left\{ \sum_{i=1}^m [p_i^-(s) - p_i^+(s)]\, x(s) f_i(x(s)) + q(s) \right\} ds\, ds_{n-1} \cdots ds_1 .$$

Let $t = a$ so that $x(a) = 0$. Hence, equation (6) implies

$$x(t_0) + \int_{t_0}^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \sum_{i=1}^m p_i^-(s)\, x(s) f_i(x(s))\, ds\, ds_{n-1} \cdots ds_1$$
$$= \int_{t_0}^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, x(s) f_i(x(s)) - q(s) \right\} ds\, ds_{n-1} \cdots ds_1 .$$

Without loss of generality, we may assume that $x(t) \geq 0$, $t \in [b, a]$. Thus, it follows from condition (iii) and (2) that

$$x(t_0) \leq \int_{t_0}^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, x(s) f_i(x(s)) - q(s) \right\} ds\, ds_{n-1} \cdots ds_1$$

$\Rightarrow$

$$1 < \int_{t_0}^{a} r_1(s_1) \int_{\alpha_1}^{s_1} r_2(s_2) \cdots \int_{\alpha_{n-1}}^{s_{n-1}} \left\{ \sum_{i=1}^m p_i^+(s)\, K_i - \frac{q(s)}{M} \right\} ds\, ds_{n-1} \cdots ds_1 .$$

This proves (4). Similarly we can prove (3), except that in (6), we now replace t by b. The sum of (3) and (4) yields (5).

We shall now clarify the importance of Theorem 1 by applying it to the particular cases:

(A) for some integer k, $1 \leq k \leq n-1$, we have

$$r_j(t) = \begin{cases} 1 & j \neq n-k \\ r(t) & j = n-k \end{cases}$$

i.e., we consider the differential equation

$$[r(t)\, x^{(n-k)}(t)]^{(k)} + \sum_{i=1}^{m} p_i(t)\, x(t)\, f_i(x(t)) = q(t)\,. \tag{7}$$

(B) $r_1(t) = r_2(t) = \cdots = r_n(t) = 1$, i.e., we consider the differential equation

$$x^{(n)}(t) + \sum_{i=1}^{m} p_i(t)\, x(t) f_i(x(t)) = q(t)\,. \tag{8}$$

COROLLARY 1. *Let the conditions of Theorem 1 hold for equation* (7). *Then*

$$(A_1) \qquad 1 < \int_b^{t_0} \frac{(t-b)^{n-k-1}}{(n-k-1)!\, r(t)} \int_t^{a_{n-k}} \frac{(s-t)^{k-1}}{(k-1)!} \left\{ \sum_{i=1}^{m} p_i^+(s)\, K_i + \frac{|q(s)|}{M} \right\} ds\, dt\,,$$

$$(A_2) \qquad 1 < \int_b^{a} \frac{(t-t_0)^{n-k-1}}{(n-k-1)!\, r(t)} \int_t^{a_{n-k}} \frac{(s-t)^{k-1}}{(k-1)!} \left\{ \sum_{i=1}^{m} p_i^+(s)\, K_i + \frac{|q(s)|}{M} \right\} ds\, dt\,,$$

$$(A_3) \qquad 2 < \int_b^{a} \frac{(t-b)^{n-k-1}}{(n-k-1)!\, r(t)} \int_t^{a_{n-k}} \frac{(s-t)^{k-1}}{(k-1)!} \left\{ \sum_{i=1}^{m} p_i^+(s)\, K_i + \frac{|q(s)|}{M} \right\} ds\, dt\,.$$

COROLLARY 2. *Let the conditions of Theorem 1 hold for equation* (8). *Then*

$$(B_1) \qquad \frac{1}{t_0-b} < \int_b^{t_0} \frac{(t-b)^{n-2}}{(n-1)!} \left\{ \sum_{i=1}^{m} p_i^+(t)\, K_i + \frac{|q(t)|}{M} \right\} dt\,.$$

$$(B_2) \qquad \frac{1}{a-t_0} < \int_{t_0}^{a} \frac{(t-t_0)^{n-2}}{(n-1)!} \left\{ \sum_{i=1}^{m} p_i^+(t)\, K_i + \frac{|q(t)|}{M} \right\} dt\,,$$

$$(B^k) \qquad \frac{a-b}{(a-t_0)(t_0-b)} < \int_b^{a} \frac{(t-b)^{n-2}}{(n-1)!} \left\{ \sum_{i=1}^{m} p_i^+(t)\, K_i + \frac{|q(t)|}{M} \right\} dt\,.$$

Remark 1. Taking $n = 2$, $m = 1$, $f_1(x(t)) = 1$ and $q(t) = 0$, then Patula's result [4] is a special case of Corollary 2.

Remark 2. Since

$$\frac{a-b}{(a-t_0)(t_0-b)} \geq \frac{4}{a-b},$$

(B_3) implies

$$\frac{4}{a-b} < \int_b^a \frac{(t-b)^{n-2}}{(n-1)!} \left\{ \sum_{i=1}^m p_i^+(t)\,K_i + \frac{|q(t)|}{M} \right\} dt$$

which is a Lyapunov inequality.

THEOREM 2. *Suppose that*

$$t^{n-2} \sum_{i=1}^m p_i^+(t) \in L^p[0,\infty) \quad , \quad t^{n-2}|g(t)| \in L^p[0,\infty), \qquad 1 \leq p < \infty.$$

If $x(t)$ *is any oscillatory solution of* (8), *then the distance between consecutive zeros of* $x(t)$ *tends to infinity as* $t \to \infty$.

Proof. Assume that the above statement is not true. Then there exists a solution $x(t)$ with its sequence of zeros $\{t_n\}$, which has a subsequence $\{t_{n_\varkappa}\}$ such that $|t_{n_{\varkappa+1}} - t_{n_\varkappa}| \leq \eta < \infty$ for any $\varkappa$. Let $s_{n_\varkappa}$ be a point in $(t_{n_\varkappa}, t_{n_{\varkappa+1}})$ where $|x(t)|$ is maximized, and put $M_{s_\varkappa} = |x(s_{n_\varkappa})|$. Then $|s_{n_\varkappa} - t_{n_\varkappa}| < \eta$, for all $\varkappa$. Since $t^{n-2} \sum_{i=1}^m p_i^+(t) \in L^p[0,\infty)$, $t^{n-2}|g(t)| \in L^p[0,\infty)$, $1 \leq p < \infty$, choose $\varkappa$ so large that

$$\left(\int_{t_{n_\varkappa}}^{\infty} \left[\frac{(t-t_{n_\varkappa})^{n-2}}{(n-1)!} \left(\sum_{i=1}^m p_i^+(t)\,\overline{K}_i + \frac{|q(t)|}{M_{ni}} \right) \right]^p dt \right)^{1/p} \leq \eta^{-1-(1/r)}$$

where $1/p + 1/r = 1$ and $\overline{K}_i = \max_{y \in [-M_{n_i}, M_{n_i}]} f_i(y)$.

From Corollary 2 (B_1), we have

$$1 < (s_{n_\varkappa} - t_{n_\varkappa}) \int_{t_{n_\varkappa}}^{s_{n_\varkappa}} \frac{(t-t_{n_\varkappa})^{n-2}}{(n-1)!} \left[\sum_{i=1}^m p_i^+(t)\,\overline{K}_i + \frac{|q(t)|}{M_{ni}} \right] dt <$$

$$< (s_{n_\varkappa} - t_{n_\varkappa}) \int_{t_{n_\varkappa}}^{s_{n_\varkappa}} \left\{ \frac{(t-t_{n_\varkappa})^{n-2}}{(n-1)!} \left[\sum_{i=1}^m p_i^+(t)\,\overline{K}_i + \frac{|q(t)|}{M_{ni}} \right] \right\}^p dt \Big)^{1/p} (s_{n_\varkappa} - t_{n_\varkappa})^{1/r}$$

$$< (s_{n_\varkappa} - t_{n_\varkappa})^{1+(1/r)}\, \eta^{-1-(1/r)} < \eta^{1+(1/r)}\, \eta^{-1-(1/r)} = 1.$$

This leads to a contradiction.

Remark 3. Taking $n = 2$, $m = 1$, $f_1(x(t)) = 1$ and $q(t) = 0$, then Patula's result [4] is a special case of Theorem 2.

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