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On the oscillation and asymptotic properties for general nonlinear differential equations

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Equazioni differenziali ordinarie. — *On the oscillation and asymptotic properties for general nonlinear differential equations* (*). Nota (**) di LU-SAN CHEN, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore trova alcune condizioni sufficienti che assicurano $\lim_{t\to\infty} x(t) = 0$ per le soluzioni oscillatorie dell'equazione

$$[r_{n-1}(t)[r_{n-2}(t)[\cdots[r_2(t)[r_1(t)h(x'(t))]']'\cdots]']' + a(t)x(t)f(x(t-g(t))) = b(t).$$

Nel caso $b(t) = 0$ l'Autore prova che con le stesse condizioni l'equazione non possiede soluzioni oscillatorie.

INTRODUCTION

The purpose of this note is to consider the general nonlinear n-th order ($n > 1$) differential equation

$$(1)\qquad [r_{n-1}(t)[r_{n-2}(t)[\cdots[r_2(t)[r_1(t)h(x'(t))]']'\cdots]']' + a(t)x(t)f(x(t-g(t))) = b(t),$$

where the functions $r_i(t)$, ($i = 1, 2, \cdots, n-1$) are positive at least on $[\tau, \infty)$. We shall consider only those solutions of the equation (1) which exist on some half-line $[t_\xi, \infty)$, where t_ξ may depend on the particular solution, and are nontrivial in any neighborhood of infinity. Such a solution is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *non-oscillatory*. Here we shall give some conditions to ensure that $\lim_{t\to\infty} x(t) = 0$ for all oscillatory solutions of the equation (1). However for $b(t) = 0$, the same conditions guarantee that all eventually nontrivial solutions of the differential equation

$$(2)\qquad [r_{n-1}(t)[r_{n-2}(t)[\cdots[r_2(t)[r_1(t)h(x'(t))]']'\cdots]']' + a(t)x(t)f(x(t-g(t))) = 0$$

are non-oscillatory.

The technique used is an adaptation of that of the Author [1] which concerns the particular case $r_1(t) = r(t)$ and $r_2(t) = r_3(t) = \cdots = r_{n-1}(t) = 1$. Recently, Staikos and Philos [2] discussed the similar problem for the following equation

$$[r_{n-1}(t)[r_{n-2}(t)[\cdots[r_1(t)[r_0(t)x(t)]']'\cdots]']' + a(t)x(t) = b(t),$$

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also, Singh [3] has treated the special case

$$r_1(t) = r(t) \quad , \quad r_2(t) = r_3(t) = \cdots = r_{n-1}(t) = 1 \quad , \quad h(x'(t)) = x'(t)$$

and

$$f(x(t - g(t))) = 1 \quad \text{in} \quad (1).$$

In addition the following assumptions will be made for the rest of this note.

Assumptions

(i) *$a(t)$, $b(t)$ and $r_i(t)$ $(i = 1, 2, \cdots, n-1)$ are continuous real-valued functions on $[\tau, \infty)$,*

(ii) *$g(t)$ is continuous, positive and bounded so that there exists some positive constant m such that*

$$0 < g(t) \leq m,$$

(iii) *$h(y)$ is continuously differentiable on $(-\infty, \infty)$ and is an odd function such that* $\operatorname{sgn} h(y) = \operatorname{sgn} y$, *there exists* $\beta > 0$, $0 < \dfrac{y}{h(y)} \leq \beta$ *and* $\lim\limits_{y\to\infty} \dfrac{y}{h(y)}$ *exists finitely so that* $\dfrac{y}{h(y)}$ *is continuously differentiable on* $[\tau, \infty)$,

(iv) *$f(y)$ is a continuous even real positive function on $(-\infty, \infty)$ and increasing on $[\tau, \infty)$ with $f(0) = 0$.*

2. Main results

THEOREM 1. *Assume that $f(y)$ is bounded and*

$$\liminf_{t\to\infty} r_1(t) > 0, \tag{3}$$

$$\int^{\infty} \frac{dt}{r_1(t)} < \infty, \tag{4}$$

$$\int^{\infty} \frac{1}{r_2(s_1)} \int_{s_1}^{\infty} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\infty} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\infty} |a(s)| \, ds \, ds_{n-2} \cdots ds_1 < \infty, \tag{5}$$

and

$$\int^{\infty} \frac{1}{r_2(s_1)} \int_{s_1}^{\infty} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\infty} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\infty} |b(s)| \, ds \, ds_{n-2} \cdots ds_1 < \infty. \tag{6}$$

Then for every oscillatory solution $x(t)$ *of the equation* (1),

$$\lim_{t\to\infty} x(t) = 0 .$$

Proof. Let

$$M_1 = \sup_{0\le y<\infty} f(y) , \tag{7}$$

and suppose on the contrary that $\lim_{t\to\infty} x(t) \neq 0$, then

$$\liminf_{t\to\infty} |x(t)| = 0 , \tag{8}$$

and for some positive d,

$$\limsup_{t\to\infty} |x(t)| > 2\,d . \tag{9}$$

From condition (3), there exists a constant $k > 0$ with $r_1(t) \ge k$ for every $t \ge t_0$. Thus, by conditions (5) and (6), we have for some $T \ge t_0$,

$$\int_T^\infty \frac{1}{r_2(s_1)} \int_{s_1}^\infty \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^\infty \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^\infty |a(s)|\, ds\, ds_{n-2} \cdots ds_1 \le \frac{1}{M_1} , \tag{10}$$

$$\int_T^\infty \frac{1}{r_2(s_1)} \int_{s_1}^\infty \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^\infty \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^\infty |b(s)|\, ds\, ds_{n-2} \cdots ds_1 < d , \tag{11}$$

and

$$\int_T^\infty \frac{dt}{r_1(t)} < \frac{1}{\beta} . \tag{12}$$

Let $T < t_1 < \alpha_{n-2} < \cdots < \alpha_3 < \alpha_2 < \alpha_1 < T_0$ be certain points where

$$x(t_1) = 0 \tag{13}$$

$$\begin{cases} [r_1(\alpha_1)\, h(x'(\alpha_1))]' = 0 \ , \quad [r_2(\alpha_2)\,[r_1(\alpha_2)\, h(x'(\alpha_2))]']' = 0 , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [r_{n-2}(\alpha_{n-2})\,[r_{n-3}(\alpha_{n-2})\,[\cdots[r_1(\alpha_{n-2})\, h(x'(\alpha_{n-2}))]'\cdots]']' = 0 , \end{cases} \tag{14}$$

and T_0 is such that

$$M = \sup_{t_1\le t\le T_0} |x(t)| > d. \tag{15}$$

Let $t_2 > T_0$ be another zero of $x(t)$ and let

$$M_0 = \sup_{t_1 \le t \le t_2} |x(t)|$$

then $M_0 > d$. Now, repeated integration from equation (1) gives

(16) $$(-1)^{n-2} [r_1(t) h(x'(t))]' + \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_{n-3}} \frac{1}{r_{n-1}(s_{n-2})} \cdot \int_{s_{n-2}}^{\alpha_{n-2}} a(s)\, x(s) f(x(s - g(s)))\, ds\, ds_{n-2} \cdots ds_2$$
$$= \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_{n-3}} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_{n-2}} b(s)\, ds\, ds_{n-2} \cdots ds_2 .$$

Since $\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_{n-2}$, we obtain from (16),

(17) $$|[r_1(t) h(x'(t))]'| \le \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \cdot \int_{s_{n-2}}^{\alpha_1} |a(s)|\, |x(s)|\, |f(x(s - g(s)))|\, ds\, ds_{n-2} \cdots ds_2$$
$$+ \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} |b(s)|\, ds\, ds_{n-2} \cdots ds_2 .$$

Let

(18) $$M = |x(t_0)|, \quad t_0 \in [t_1, t_2],$$

then

$$\pm M = x(t_0) = \int_{t_1}^{t_0} x'(t)\, dt$$

which implies

(19) $$M \le \int_{t_1}^{t_0} |x'(t)|\, dt.$$

Similarly

(20) $$M \le \int_{t_0}^{t_2} |x'(t)|\, dt.$$

From (19) and (20) we have

$$2M \le \int_{t_1}^{t_2} |x'(t)|\, dt.$$

By Schwarz's inequality we get

$$(21)\qquad 4M^2 \leq \int_{t_1}^{t_2} \frac{x'(t)}{h(x'(t))}\frac{dt}{r_1(t)} \int_{t_1}^{t_2} [r_1(t)\, h(x'(t))]\, x'(t)\, dt\,.$$

since $\dfrac{x'(t)}{h(x'(t))}$ is continuous and positive.

Therefore

$$4M^2 \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \int_{t_1}^{t_2} [r_1(t)\, h(x'(t))]\, x'(t)\, dt$$

since $0 < \dfrac{x'(t)}{h(x'(t))} \leq \beta$.

Integrating the second integral of the right hand side by parts we obtain

$$(22)\qquad 4M \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \int_{t_1}^{t_2} |\, [r_1(t)\, h(x'(t))]'\, |\, dt$$

since $x(t_1) = x(t_2) = 0$. from (18) and (22), we get

$$(23)\qquad 4M \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \left\{ \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} |\, a(s)\,|\,|\, x(s)\,|\, f(L)\, ds\, ds_{n-2} \cdots ds_1 \right.$$

$$\left. + \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} |\, b(s)\,|\, ds\, ds_{n-2} \cdots ds_1 \right\},$$

where $L = \sup\{x(t) \mid t \in (t_1 - m, t_2),\, t_1, t_2 > m\}$.

Dividing by M and noting that $t_2 > \alpha_1$ we have from (23),

$$(24)\qquad 4 \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \left\{ f(L) \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} |\, a(s)\,|\, ds\, ds_{n-2} \cdots ds_1 \right.$$

$$\left. + \frac{1}{M} \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} |\, b(s)\,|\, ds\, ds_{n-2} \cdots ds_1 \right\}.$$

Then, from (7) and (24), we have

$$
(25) \qquad 4 \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \left\{ M_1 \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} | a(s) | \, ds \, ds_{n-2} \cdots ds_1 + \frac{1}{M_0} \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} | b(s) | \, ds \, ds_{n-2} \cdots ds_1 \right\}.
$$

From (10), (11), (12), the fact that $M_0 > d$ and (25), we have

$$
(26) \qquad 4 \leq 1 + \frac{d}{d} = 2 .
$$

This contradiction proves the theorem.

THEOREM 2. *Suppose that* (3), (4) *and* (5) *are satisfied and that* $f(y)$ *is bounded. Then every nontrivial solution of* (2) *is non-oscillatory.*

Proof. Following the proof of Theorem 1, we arrive at conclusion (25). From (25) we obtain

$$
(27) \qquad 4 \leq \beta \int_{t_1}^{t_2} \frac{dt}{r_1(t)} \left\{ M_1 \int_{t_1}^{t_2} \frac{1}{r_2(s_1)} \int_{s_1}^{\alpha_1} \frac{1}{r_3(s_2)} \cdots \int_{s_{n-3}}^{\alpha_1} \frac{1}{r_{n-1}(s_{n-2})} \int_{s_{n-2}}^{\alpha_1} | a(s) | \, ds \, ds_{n-2} \cdots ds_1 \right\} \leq 1 ,
$$

from (10) and (12). This contradiction proves the theorem.

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