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A generalization of a theorem of Reissig for a certain non-autonomous differential equation

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Equazioni differenziali ordinarie. — *A generalization of a theorem of Reissig for a certain non-autonomous differential equation.* Nota (*) di RAINER ANSORGE e BAHMAN MEHRI, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota è generalizzato un teorema di R. Reissig sull'esistenza di una soluzione periodica dell'equazione differenziale non autonoma

$$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f(t, x) = e(t).$$

1. INTRODUCTION. In the paper [1] Reissig investigated the existence of periodic solutions of the equations

$$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f(x) = e(t) \tag{1}$$

where a_i $(i = 1, 2, \cdots, n)$ are real positive constants and the assumption on f is such that $|f(x)| \leq F$ (a constant) for all x. This assumption is very strong and its applications are very limited.

The object of the present Note is to extend Reissig's result further, i.e. we consider the equation

$$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f(t, x) = e(t), \quad a_n \neq 0 \tag{2}$$

where the functions $f(t, x)$ and $e(t)$ are continuous real valued functions, periodic with respect to t of period ω,

i.e. $f(t + \omega, x) = f(t, x)$, $e(t + \omega) = e(t)$ and $\int_0^\omega e(t)\,dt = 0$

($f(t, x)$ not necessarily bounded for all x).

We further assume that the n-th degree polynomial $P_n(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ has n-distinct roots $\lambda_i \neq 0$, $i = 1, 2, \cdots, n$, and all solutions of initial value problems for (2) extend to $[0, \omega]$.

It will be shown under some conditions on f: There exists at least one solution of (2) satisfying the periodic boundary conditions

$$x^{(i)}(0) = x^{(i)}(\omega), \qquad i = 0, 1, 2, \cdots, n. \tag{3}$$

The method which is used here is similar to Lazer's one [3].

2. In this section $P_n(\lambda)$ designates a polynomial of degree n (n arbitrary), for which the coefficient of λ^n equals 1.

(*) Pervenuta all'Accademia il 6 settembre 1976.

LEMMA 1. *If* $P_n(\lambda)$ *is a polynomial of degree* n *with distinct roots* $\lambda_i \neq 0$, $i = 1, 2, \cdots, n$, *then for* $n \geq 2$

$$\sum_{i=1}^{n} \frac{1}{P_n'(\lambda_i)} = 0 \tag{4}$$

and

$$\sum_{i=1}^{n} \frac{1}{\lambda_i P_n'(\lambda_i)} = \frac{(-1)^{n-1}}{\lambda_1 \lambda_2 \cdots \lambda_n} = (-1)^{n-1} \frac{1}{\prod_{i=1}^{n} \lambda_i}. \tag{5}$$

Proof. (By Lagrange's interpolation theorem). If $f(\lambda)$ is a function with given values in n-distinct points $\lambda_1, \lambda_2, \cdots, \lambda_n$, then there exists an unique polynomial of maximal degree $n - 1$ which coincides with f at these distinct points. This polynomial is given by

$$Q_{n-1}(\lambda) = \sum_{i=1}^{n} L_i(\lambda) f(\lambda_i) \tag{6}$$

where

$$L_i(\lambda) = \frac{\prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda - \lambda_j)}{P_n'(\lambda_i)}$$

and we have $Q_{n-1}(\lambda) \equiv f(\lambda)$ if f itself is a polynomial of maximal degree $n - 1$. In particular when $f(\lambda) \equiv 1$, the coefficient of λ^{n-1} in (6) has to vanish ($n \geq 2$) and we obtain (4). The coefficient of λ^0 in (6) has to equal 1; this leads to

$$1 = \sum_{i=1}^{n} \frac{(-1)^{n-1}}{P_n'(\lambda_i)} \prod_{\substack{j=1 \\ i \neq 1}}^{n} \lambda_j = (-1)^{n-1} \prod_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \frac{1}{\lambda_i P_n'(\lambda_i)}$$

or

$$\sum_{i=1}^{n} \frac{1}{\lambda_i P_n'(\lambda_i)} = (-1)^{n-1} \frac{1}{\prod_{i=1}^{n} \lambda_i}$$

which completes the proof of Lemma 1.

LEMMA 2. *If* $P_n(\lambda)$ *satisfies the assumptions of Lemma 1, then*

$$\sum_{i=1}^{n} \frac{\lambda_i^j}{P_n'(\lambda_i)} = 0, \qquad j = 0, 1, 2, \cdots, n-2. \tag{7}$$

Proof. We proceed by induction: (7) is already proved for $j = 0$ (see (4)). Now assuming that (7) is true for any $j = K \leq n - 3$, then

$$0 = \sum_{i=1}^{n-1} \frac{\lambda_i^K}{P_{n-1}'(\lambda_i)} = \sum_{i=1}^{n-1} \frac{\lambda_i^{K+1}}{P_n'(\lambda_i)} - \lambda_n \sum_{i=1}^{n-1} \left(\frac{\lambda_i^K}{P_n'(\lambda_i)} - \frac{\lambda_n^{K+1}}{P_n'(\lambda_i)} + \frac{\lambda_n^{K+1}}{P_n'(\lambda_i)} \right) =$$

$$= \sum_{i=1}^{n} \frac{\lambda_i^{K+1}}{P_n'(\lambda_i)} - \lambda_n \sum_{i=1}^{n} \frac{\lambda_i^K}{P_n'(\lambda_i)} = \sum_{i=1}^{n} \frac{\lambda_i^{K+1}}{P_n'(\lambda_i)}.$$

(Note: $P_n'(\lambda_i) = (\lambda_i - \lambda_n) P_{n-1}'(\lambda_i)$). Hence Lemma 2 is proven.

LEMMA 3. *With the assumptions of Lemma 1, we have*

$$\sum_{i=1}^{n} \frac{\lambda_i^{n-1}}{P_n'(\lambda_i)} = 1 . \tag{8}$$

Proof. Assume $n = 2$, then

$$\sum_{i=1}^{2} \frac{\lambda_i}{P_2'(\lambda_i)} = -\frac{\lambda_1}{\lambda_2 - \lambda_1} + \frac{\lambda_2}{\lambda_2 - \lambda_1} = 1$$

which is true. Now assume (8) is true for $n-1$, i.e.

$$1 = \sum_{i=1}^{n-1} \frac{\lambda_i^{n-2}}{P_{n-1}'(\lambda_i)} = \sum_{i=1}^{n-1} \frac{\lambda_i^{n-1}}{P_n'(\lambda_i)} - \lambda_n \sum_{i=1}^{n-1} \frac{\lambda_i^{n-2}}{P_n'(\lambda_i)} + \frac{\lambda_n^{n-1}}{P_n'(\lambda_n)} - \lambda_n \frac{\lambda_n^{n-2}}{P_n'(\lambda_n)} =$$

$$= \sum_{i=1}^{n} \frac{\lambda_i^{n-1}}{P_n'(\lambda_i)} - \lambda_n \sum_{i=1}^{n} \frac{\lambda_i^{n-2}}{P_n'(\lambda_i)} = \sum_{i=1}^{n} \frac{\lambda_i^{n-1}}{P_n'(\lambda_i)}$$

which completes the proof of Lemma 3.

3. GREEN'S FUNCTION. We can define the Green's function for the equation

$$y^{(n+1)} + a_1 y^{(n)} + \cdots + a_n y' = \frac{1}{\omega} \tag{9}$$

with the periodic boundary conditions

$$y^{(i)}(0) = y^{(i)}(\omega) , \qquad i = 0 , 1 , 2 , \cdots , n$$

as

$$G(t,s) = \begin{cases} \dfrac{1}{2a_n} + \displaystyle\sum_{j=1}^{n} \frac{e^{\lambda_j(\omega - s + t)}}{\lambda_j P_n'(\lambda_j)(e^{\lambda_j \omega} - 1)} + \dfrac{t}{a_n \omega} , & 0 \leqq t < s \leqq \omega \\ \dfrac{-1}{2a_n} + \displaystyle\sum_{j=1}^{n} \frac{e^{\lambda_j(t-s)}}{\lambda_j P_n'(\lambda_j)(e^{\lambda_j \omega} - 1)} + \dfrac{t}{a_n \omega} , & 0 \leqq s < t \leqq \omega , \end{cases}$$

provided that $e^{\lambda_j \omega} \neq 1$, $j = 1 , 2 , \cdots , n$. Then using Lemmas 1-3, we can easily show that indeed

a) $G(t,s)$ is continuous with its derivatives up to order $n-1$ on $[0,\omega] \times [0,\omega]$, and furthermore there exist constants M_1 and M_2 such that $|G(t,s)| \leq M_1$, $|G_t(t,s)| \leq M_2$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.

b) $\dfrac{\partial^n}{\partial t^n} G(t, t-) - \dfrac{\partial^n}{\partial t^n} G(t, t+) = -1$

c) $\dfrac{\partial^i}{\partial t^i} G(t,s) \Big|_{t=0} = \dfrac{\partial^i}{\partial t^i} G(t,s) \Big|_{t=\omega}$, $\qquad i = 0 , 1 , 2 , \cdots , n .$

4. MAIN THEOREM. *Assume that the following conditions hold:*

i) $xf(t, x) \geq 0$ *for* $|x| \geq b$ *with some non-negative real number* b, *and for all* t.

ii) *There exists a constant* D, *such that*

$$b + 3m \leq D \tag{10}$$

where

$$m = \max \{M, \omega M_1 (M + E)\}$$

and

$$M = \max \{|f(t, x)| : t \in [0, \omega], |x| \leq D\} \quad , \quad E = \max \{|e(t)| : t \in [0, \omega]\}.$$

Then equation (2) *has at least one solution* $x(t)$ *satisfying the periodic boundary conditions* (3).

Proof. Let us consider the following integral equation

$$x(t) = \int_0^\omega G(t, s) \{f(s, x(s)) - e(s)\} ds. \tag{11}$$

It follows that $x(t)$ defined as in (11) satisfies the periodic boundary conditions (3), and moreover

$$x^{(n+1)} + a_1 x^{(n)} + \cdots + a_n x' + f(t, x) = e(t) + \frac{1}{\omega} \int_0^\omega f(s, x(s)) ds.$$

In what follows, we shall prove that (11) has a solution, say $\varphi(t)$, such that

$$\int_0^\omega f(s, \varphi(s)) ds = 0.$$

Let S be the space of all continuous functions on $[0, \omega]$. If $\theta \in S$, let $\|\theta\| = \max |\theta(t)|, t \in [0, \omega]$, and let R denote the real numbers, and let $B = S \times R$. If $(\theta, a), (\theta_1, a_1), (\theta_2, a_2) \in B, x_1, x_2 \in R$, let us define

$$|(\theta, a)| = \|\theta\| + |a|$$

$$x_1(\theta_1, a_1) + x_2(\theta_2, a_2) = (x_1 \theta_1 + x_2 \theta_2, x_1 a_1 + x_2 a_2).$$

With these definitions, B is a complete normed linear space. For each $(\theta, a) \in B$, we define

$$T[(\theta, a)] = (\theta^*, a^*)$$

where

$$(12) \qquad \theta^* = a + \int_0^\omega G(t, s)\{f(s, \theta(s)) - e(s)\}\, ds$$

$$a^* = a - \frac{1}{\omega}\int_0^\omega f(s, \theta^*, (s))\, ds.$$

Then T is a continuous mapping from B into B.

Let $K = \{(\theta, a) \in B \mid \|\theta\| \leq D, |a| \leq b + 2m\}$. In order to apply Schauder's fixed point theorem, one has to establish the following facts:

a) $T(B) \subset B$.

b) $T(B)$ has a compact closure.

To prove (*a*), from one hand we have

$$\|\theta^*\| \leq |a| + M_1 \omega (M + E) \leq b + 2m + m \leq D, \quad \text{for} \quad (\theta, a) \in B,$$

and on the other and if $-(b + m) \leq a \leq (b + m)$, since $\|\theta^*\| \leq D$ it follows that

$$\left| \frac{1}{\omega}\int_0^\omega f(s, \theta^*(s))\, ds \right| \leq M \leq m;$$

consequently one obtains

$$(13) \qquad -(b + 2m) \leq a - \frac{1}{\omega}\int_0^\omega f(s, \theta^*(s))\, ds \leq b + 2m.$$

Now, by considering the inequality

$$\|\theta^* - a\| \leq M \leq m,$$

the condition $a \geq b + m$ implies $\theta^*(t) \geq b$ and the condition $a \leq -(b + m)$ leads to $\theta^*(t) \leq -b$ for all t.

Therefore by (i), for $b + m \leq a \leq b + 2m$, we have $f(t, \theta^*(t)) \geq 0$, from which it follows that

$$(14) \qquad b \leq a - \frac{1}{\omega}\int_0^\omega f(s, \theta^*(s))\, ds \leq b + 2m,$$

and for $-(b + 2m) \leq a \leq -(b + m)$, we have $f(t, \theta^*(t)) \leq 0$, and hence

$$(15) \qquad -(b + 2m) \leq a \leq a - \frac{1}{\omega}\int_0^\omega f(s, \theta^*(s))\, ds \leq -b.$$

Now, the assertion (*a*) follows from (13), (14) and (15).

To prove (*b*), let (θ_n^*, a_n^*) be an infinite sequence in T (B), then we have to show: There exists a subsequence $\{(\theta_{n_k}^*, a_{n_k}^*)\}$ and an element $(\theta^*, a^*) \in S \times R$ such that

$$\lim_{k \to \infty} |(\theta_{n_k}^*, a_{n_k}^*) - (\theta^*, a^*)| = 0 .$$

We know that for every $n \in \mathbf{N}$ there exists $(\theta_n, a_n) \in B$ such that $T(\theta_n, a_n) = (\theta_n^*, a_n^*)$. Consider the function

$$V_n(t) = \int_0^\omega G(t, s) \{f(s, \theta_n(s)) - e(s)\} \, ds$$

and

$$\frac{dV_n}{dt} = \int_0^\omega G_t(t, s) \{f(s, \theta_n(s)) - e(s)\} \, ds.$$

Then $\|V_n(t)\| \leq M_1 \omega (M + E) \leq m$ and $\left\| \frac{dV_n}{dt} \right\| \leq M_2 \omega (M + E) \leq \frac{M_2}{M_1} m$.

The preceding inequalities show that the sequence $\{V_n(t)\}$ is uniformly equicontinuous and is contained in a closed ball with radius m around the origin $B_m(o)$ in S. By Ascoli's Lemma, there exists a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ and a $V \in B_m(o)$ such that

$$\lim_{k \to \infty} \| V_{n_k} - V \| = 0.$$

On the other hand since $a_n \in [-(b + 2m), (b + 2m)]$ for all $n \in \mathbf{N}$, we can extract a convergence subsequence denoted by $\{a_{n_k}\}$. Clearly the subsequence $\{(\theta_{n_k}^*, a_{n_k}^*)\}$ of $\{(\theta_n^*, a_n^*)\}$ where

$$\theta_{n_k}^* = a_{n_k} + V_{n_k} \qquad \text{and} \qquad a_{n_k}^* = a_{n_k} - \frac{1}{\omega} \int_0^\omega f(s, \theta_{n_k}^*(s)) \, ds ,$$

converges to $(\theta^*, \hat{a})$, with $\theta^* = \alpha^* + V$, where α^* is the limit point of a_{n_k} and $\hat{a} = \alpha^* - \frac{1}{\omega} \int_0^\omega f(s, \theta^*(s)) \, ds$ in $S \times R$. This establishes assertion (*b*). Now by Schauder's fixed point theorem there exists at least an element $(\varphi, \gamma) \in B$ such that

$$(\varphi, \gamma) = T(\varphi, \gamma), \qquad \text{i.e.} \qquad \varphi = a + \int_0^\omega G(t, s) \{f(s, \varphi(s)) - e(s)\} \, ds$$

and $\int_0^\omega f(s, \varphi(s)) \, ds = 0$, which completes the proof of the theorem.

Remark. In case when the polynomial $P_n(\lambda)$ does not have distinct roots, then the construction of Green's function as given in section 3 is more complicated, but it is not hard to prove theoretically its existence. In particular when $n = 1$, and $P_{n+1}(\lambda) = \lambda^2$, then we can define the Green's function as follows

$$G(t, s) = \frac{1}{2\omega} \begin{cases} \left(s - t - \dfrac{\omega}{2}\right)^2 ; & 0 \leq t \leq s \leq \omega \\ \left(t - s - \dfrac{\omega}{2}\right)^2 ; & 0 \leq s \leq t \leq \omega \end{cases}$$

obviously $M_1 = \frac{\omega}{8}$, $M_2 = \frac{1}{2}$.

COROLLARY. *If in addition to all the hypotheses of our Main theorem, the function $f(t, x)$ is locally Lipschitzian with respect to x, then* (2) *has an ω-periodic solution.*

5. In this section, we shall consider some applications of our main theorem

(A_1): Consider the equation

$$x'' + x' + \beta x + x^3 = E \cos t \tag{16}$$

where E, $\beta > 0$ are real constants. We want to show that equation (16) possesses at least one periodic solution of period 2π, provided β and $|E|$ are sufficiently small. In order to do so, we must show that there exists a constant D such that Condition (10) of our main theorem is satisfied. But for the equation (16), we have $M = \max \{\beta x + x^3 : |x| \leq D\} = \beta D + D^3$ and $M_1 = \frac{3\pi + e^{2\pi}}{2\pi}$, which implies $m = (3\pi + e^{2\pi})(\beta D + D^3 + |E|)$.

Therefore condition (10) is satisfied, if there exists a constant D such that

$$\beta D + D^3 + |E| \leq \frac{1}{3(3\pi + e^{2\pi})} D$$

or

$$D^3 + |E| \leq \left(\frac{1}{3(3\pi + e^{2\pi})} - \beta\right) D.$$

It is obvious, for $0 \leq \beta < \frac{1}{3(3\pi + e^{2\pi})}$ and $|E|$ sufficiently small, that such a D exists.

(A_2): Consider the equation

$$x''' + cx' + x^3 = E \cos t \tag{18}$$

where E, $c > 0$ are real constants. We want to show that equation (18) possesses a periodic solution of period 2π, provided $2\sqrt{c} \leq 1$, and $|E|$ is sufficiently small. It follows from section 3 that for the equation (18), we can define the Green's function as:

$$G(t, s) = \frac{1}{c} \begin{cases} \dfrac{1}{2} - \dfrac{\sin \sqrt{c}\,(\pi - s + t)}{2 \sin \sqrt{c}\,\pi} + \dfrac{t}{2\pi}; & 0 \leq t \leq s \leq 2\pi \\ -\dfrac{1}{2} + \dfrac{\sin \sqrt{c}\,(\pi + s - t)}{2 \sin \sqrt{c}\,\pi} + \dfrac{t}{2\pi}; & 0 \leq s \leq t \leq 2\pi . \end{cases}$$

Now, since $\sqrt{c} \leq \frac{1}{2}$, it follows $\sin \sqrt{c}\,\pi \geq 2\sqrt{c}$, which implies $M_1 = \dfrac{6\sqrt{c} + 1}{4c\sqrt{c}}$ and $m = 2\pi \cdot \dfrac{6\sqrt{c}+1}{4c\sqrt{c}} (D^3 + |E|)$. The condition (10) is satisfied if there exists a constant D, such that

$$D^3 + |E| \leq \frac{2c\sqrt{c}}{3\pi(1 + 6\sqrt{c})} D .$$

Now, if $|E|$ is sufficiently small, it is obvious that such a D exists.

(A_3): Assume, instead of assumption (ii) of our main theorem,
(ii)′ $\dfrac{f(t, x)}{x} \to 0$ as $|x| \to \infty$ uniformly in t.

Then equation (2) possesses a periodic solution of period ω. In order to prove this, we have to show that there exists a constant D such that condition (10) of our main theorem is satisfied.

But condition (ii)′ implies, for any $\varepsilon > 0$, that there exists a number $L(\varepsilon)$ such that

$$|f(t, x)| < \varepsilon D \qquad \text{if} \quad D > L(\varepsilon) \qquad \text{and} \quad |x| \leq D .$$

Now, assuming

$$0 < \delta < \min\left\{\frac{1}{3}, \frac{1}{3 M_1 \omega}\right\},$$

$$D = \max\left\{\frac{b}{1 - 3\delta}, \frac{b + 3 M_1 \|e\| \omega}{1 - 3 M_1 \delta\omega}, L(\delta)\right\}$$

and

$$m = \max\{\delta D, M_1(\delta D + \|e\|)\omega\},$$

it follows that $b + 3m \leq D$, i.e. the condition (10) of our main theorem is satisfied.

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