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On surjectivity for e—quasibounded multivalued maps

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Analisi funzionale. — *On surjectivity for e–quasibounded multi-valued maps* (*). Nota (**) di GIUSEPPE CONTI e PAOLO NISTRI, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni di suriettività per applicazioni multivoche in spazi di Banach.

1. INTRODUCTION

Let E be a Banach space and $T : E \multimap E$ be a multi-valued map. The purpose of this Note is to investigate conditions under which the map $I - T$ is onto, where I is the identity on E.

We obtain results which extend Theorems proved in [4], [9] and [11]. The main tool we have used in this paper is a suitable definition of asymptotic spectrum of a multi-valued map $T : E \multimap E$ as introduced in [1].

2. NOTATIONS AND DEFINITIONS

Let E , F be two real or complex Banach spaces and $T : E \multimap F$ be a multi-valued map. We recall that T is *upper semicontinuous* on E if, for any $x \in E$, $T(x)$ is compact (and not empty) and, for any open set U containing $T(x)$, there exists a neighborhood V of x such that $T(y) \subset U$ for all $y \in V$.

If for any $x \in E$ the set $T(x)$ is acyclic in the Vietoris homology theory with coefficients in Q, then T is said to be *acyclic-valued*.

Let A be any bounded set of E. Denote by $\alpha(A)$ the Kuratowski measure of non-compactness of A (see [7]). If there exists $k \geq 0$ such that $\alpha(T(A)) \leq \leq k\alpha(A)$ for any bounded set $A \subset E$, then T is called an *α-Lipschitz* map with constant k. According to [2] we shall say that T is *α-contractive* (*α-nonexpansive*) if $k < 1$ ($k = 1$).

If $\alpha(T(A)) < \alpha(A)$ for any bounded and non precompact set $A \subset E$, then T is said to be *condensing*.

Let $M \subset E$. Define $\Phi(M) = \sup\{\|y\| : y \in M\}$.

The upper semicontinuous map $T : E \multimap F$ is *quasibounded* if

$$|T| = \limsup_{\|x\| \to +\infty} \frac{\Phi(T(x))}{\|x\|} < +\infty .$$

The number $|T|$ is called the *quasinorm* of T (see [5]).

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The set Q (E) of all the multi-valued quasibounded maps $T : E \multimap E$ cannot be endowed with the structure of a vector space since, for any $T \in Q(E)$, we have that $T - T = 0$ if and only if T is single-valued.

Let $T, S \in Q(E)$. We say that T satisfies *property* (A) if

$$\lim_{\|x\| \to +\infty} \frac{\operatorname{diam} T(x)}{\|x\|} = 0$$

where diam T (x) is the diameter of T (x).

If $\lim_{\|x\| \to +\infty} \frac{\delta(T(x), S(x))}{\|x\|} = 0$ then T and S are said to be *asymptotically equivalent* $(T \sim S)$, where δ is the Hausdorff distance between T (x) and S (x). In the case when S is linear then T is *asymptotically linear* and $S = T'_\infty$ is said to be the *asymptotic derivative* of T (see [1]).

Clearly an asymptotically linear multi-valued map $T : E \multimap E$ satisfies property (A).

Let $D_r = \{x \in E : \|x\| \leq r\}$. A multi-valued map T satisfies the *Leray-Schauder condition* on D_r if $\lambda x \in T(x)$ for some $x \in S_r$, the boundary of D_r, implies that $\lambda \leq 1$.

3. Some surjectivity theorems for *e*-quasibounded maps

Let $T : E \multimap E$ be a multi-valued upper semicontinuous map. We say that the real number μ is an *eigenvalue* of T if there exists $x \in E$, $x \neq 0$, such that $s\,x \in T(x)$ for some $s \in \mathbf{R}$ and

$$\mu = \sup \{s \in \mathbf{R} : sx \in T(x)\}.$$

Then x is said to be an *eigenvector* of T belonging to μ.

Let $r > 0$. Consider the extended real number defined as follows $b(r, T) = \sup \{\mu \geq 0 : \mu$ is an eigenvalue of T with eigenvector x such that $\|x\| = r\}$ if the above set is not empty, and $b(r, T) = 0$ otherwise.

For any $p \in E$ we define

1) $(T)_0 = \inf \{b(r, T) ; r > 0\}$.

2) $(T)_p = (T + p)_0$.

3) $(T) = \sup \{(T)_p ; p \in E\}$.

If $(T) < +\infty$, the mapping T is said to be *essentially-quasibounded* (*e-quasibounded*) and (T) will be called the *essential quasinorm* (*e-quasinorm*) of the map T.

We have the following

LEMMA 1

(i) $(\nu T)_0 = \nu (T)_0 \quad \nu \in \mathbf{R}^+$

(ii) $(\nu T) = \nu (T) \quad \nu \in \mathbf{R}^+$

(iii) *If* T *is quasibounded then* T *is e-quasibounded and* $(T) \leq |T|$.

Proof. The first property follows immediately from the fact that ρ is an eigenvalue of T if and only if $\nu\rho$ is an eigenvalue of νT ($\nu > 0$).
Definition 2) and (i) imply that

$$(\nu T)_p = (\nu T + p)_0 = (\nu (T + \nu^{-1} p))_0 = \nu (T + \nu^{-1} p)_0 = \nu (T)_{\nu^{-1} p}$$

Hence $(\nu T) = \nu (T)$.

In order to prove (iii) observe that

$$b(r, T) \leq \sup_{\|x\| \geq r} \frac{\Phi(T(x))}{\|x\|}.$$

For any $p \in E$ we have

$$(T)_p = (T + p)_0 \leq \inf_{r>0} \sup_{\|x\| \geq r} \frac{\Phi(T(x) + p)}{\|x\|} =$$

$$= \lim_{\|x\| \to +\infty} \sup \frac{\Phi(T(x) + p)}{\|x\|} = |T + p| = |T|$$

hence $(T) = \sup_{p \in E} (T)_p \leq |T|$

Q.E.D.

Remark. There are examples of T such that $(T) < |T|$ (see [4]).

THEOREM 1. *Let* $T : E \multimap E$ *be an upper semicontinuous acyclic-valued condensing map.*

a) *If* $(T)_0 < 1$ *then* T *has a fixed point.*

b) *If* $(T)_p < 1$ *the equation* $p \in x - T(x)$ *has a solution.*

c) *If* $(T) < 1$ *then* $I - T$ *is surjective* (i.e. $(I - T)(E) = E$).

Proof. It is easy to see that a) $\Rightarrow$ b) $\Rightarrow$ c), therefore it sufficies to prove a).

There exists $r > 0$ such that $b(r, t) < 1$, that is all the eigenvalues λ corresponding to eigenvectors x with $\|x\| = r$ are smaller than 1.

Therefore the mapping T satisfies the Leray-Schauder condition on the boundary of the ball D_r. Hence, by a theorem due to Martelli [8], T has at least one fixed point.

Q.E.D.

A surjectivity result proved in [9] for the class of upper semicontinuous acyclic-valued condensing maps with $|T| < 1$ can be derived from the above Theorem since $(T) \leq |T|$.

In the case of single-valued maps we obtain as corollaries some surjectivity results of Granas [5] and Dubrowskij [3] (see also [9]).

COROLLARY 1. *Let* H *be an Hilbert space and let* $T : H \multimap H$ *be an upper semicontinuous acyclic-valued condensing map.*

Define $s(T, x) = \sup \{\langle z, x\rangle : z \in T(x)\}$. *If*

$$s(T) = \limsup_{\|x\| \to +\infty} \frac{s(T, x)}{\|x\|^2} < 1$$

then $I - T$ *is onto.*

Proof. By Theorem 1 it suffices to prove that $(T) \leq s(T)$. In fact, take $a > s(T)$. Then there exists $\rho > 0$ such that $\|x\| \geq \rho$ and $z \in T(x)$ implies $\langle z, x\rangle \leq a \|x\|^2$.

The above inequality shows that there are no positive eigenvalues greater than a corresponding to eigenvectors x such that $\|x\| \geq \rho$. Hence $(T)_0 \leq a$. The arbitrarity of $a > s(T)$ implies $(T)_0 \leq s(T)$. Clearly $s(T + p) = s(T)$ for all $p \in E$. Therefore $(T + p)_0 \leq s(T)$, and so $(T) \leq s(T)$.

Q.E.D.

Notice that in the case of single-valued maps the above result is due to Krasnosel'skij (see [6]).

COROLLARY 2. *Let* $T : E \multimap E$ *be an upper semicontinuous acyclic-valued map. Assume that* T *is* α*-Lipschitz with constant* k.

Then for any υ *such that* $\upsilon > \max \{k, (T)\}$ *the map* $\upsilon I - T$ *is onto.*

Proof. Clearly $\upsilon^{-1} T$ is α-contractive and $(\upsilon^{-1} T) < 1$.

Q.E.D.

In [1] a notion of asymptotic spectrum for multi-valued quasibounded maps is given. We state now a Theorem which gives a relation between this spectrum and the concept of e-quasibounded maps.

Let us recall first some definitions of [1].

Let $T \in Q(E)$. Put

$$d(\lambda - T) = \liminf_{\|x\| \to +\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} \qquad \lambda \in \mathbf{C}$$

where $\lambda - T : E \multimap E$ is the map defined by $(\lambda - T)x = \lambda x - Tx$. The *asymptotic spectrum* $\Sigma(T)$ of T is the set

$$\Sigma(T) = \{\lambda \in \mathbf{C} : d(\lambda - T) = 0\}$$

and the *spectral radius* $r(T)$ of T is the non-negative real number

$$r(T) = \sup \{|\lambda| : \lambda \in \Sigma(T)\}.$$

Moreover we put $r^+(T) = \sup \{\lambda \geq 0 : \lambda \in \Sigma(T)\}$. ($r(T) = r^+(T) = 0$ if $\Sigma(T) = \varnothing$).

The following theorem holds

THEOREM 2. *Let* $T \in Q(E)$. *If* T *satisfies property* (A) *then* T *is e-quasibounded and* $(T) \leq r^+(T) \leq |T|$.

Proof. Clearly $r^+(T) \leq r(T)$. On the other hand $r(T) \leq |T|$ (see [1]). Therefore it is enough to prove that $(T) \leq r^+(T)$. Observe that $d(\lambda - T) = d(\lambda - T + p)$ for all $p \in E$.

In fact

$$d(\lambda - T + p) = \liminf_{\|x\| \to +\infty} \frac{\delta(T(x) - p, \lambda x)}{\|x\|} \leq$$

$$\leq \liminf_{\|x\| \to +\infty} \frac{\delta(T(x), \lambda x)}{\|x\|} + \limsup_{\|x\| \to +\infty} \frac{\delta(-p, 0)}{\|x\|} = d(\lambda - T).$$

On the other hand

$$d(\lambda - T) = \liminf_{\|x\| \to +\infty} \frac{\delta(T(x) - p, \lambda x - p)}{\|x\|} \leq$$

$$\leq \liminf_{\|x\| \to +\infty} \frac{\delta(T(x) - p, \lambda x)}{\|x\|} + \limsup_{\|x\| \to +\infty} \frac{\delta(0, -p)}{\|x\|} = d(\lambda - T + p).$$

Therefore $r^+(T + p) = r^+(T)$ for any $p \in E$. Hence it sufficies to show that $(T)_0 \leq r^+(T)$.

Assume the contrary. Then there exist $\varepsilon > 0$ and two sequences $\{\lambda_n\} \subset \mathbf{R}^+$ and $\{x_n\} \subset E, \|x_n\| \to +\infty$ as $n \to \infty$, such that $\lambda_n \geq r^+(T) + \varepsilon$ and $\lambda_n x_n \in T(x_n)$. Therefore $\|\lambda_n x_n\| \leq \delta(T(x_n), 0)$ and so

$$0 \leq \lambda_n \leq \frac{\delta(T(x_n), 0)}{\|x_n\|} \leq |T|.$$

The sequence $\{\lambda_n\}$ is bounded, hence we may assume without loss of generality that $\lambda_n \to \lambda \geq r^+(T) + \varepsilon$.

We have

$$\frac{\delta(T(x_n), \lambda x_n)}{\|x_n\|} \leq \frac{\delta(T(x_n), \lambda_n x_n)}{\|x_n\|} + |\lambda_n - \lambda| \leq \frac{\operatorname{diam} T(x_n)}{\|x_n\|} + |\lambda_n - \lambda|.$$

Thus

$$\lim_{\|x_n\| \to +\infty} \frac{\delta(\lambda x_n, T(x_n))}{\|x_n\|} = 0.$$

Then $d(\lambda - T) = 0$ that is $\lambda \in \Sigma(T)$, contradicting the hypothesis.

Q.E.D.

This Theorem extends to multi-valued maps a Theorem of M. Furi and A. Vignoli [4] (see also [11]).

We give now an example which shows that in the above Theorem the assumption $\lim\limits_{\|x\|\to+\infty} \dfrac{\operatorname{diam} T(x)}{\|x\|} = 0$ is essential.

Example. Let $T: \mathbf{R} \multimap \mathbf{R}$ be a multi-valued map defined by $T(x) = \{y \in \mathbf{R} : -|x| \leq y \leq |x|\}$. T is compact and quasibounded with $|T| = 1$.

We can prove that $(T) = 1$, while $\Sigma(T) = \varnothing$ and so $r^+(T) = 0$.

In fact

$$\lim_{\|x\|\to+\infty} \frac{\operatorname{diam} T(x)}{\|x\|} = 2 .$$

COROLLARY 3. *Let* $T: E \multimap E$ *be an upper semicontinuous condensing, acyclic-valued map. Suppose that* T *is asymptotically linear.*

If $r^+(T'_\infty) < 1$, *then* $I - T$ *is onto.*

Proof. By definition $T'_\infty \sim T$ and so $\Sigma(T) = \Sigma(T'_\infty)$ (see [1]). It follows that $r^+(T) = r^+(T'_\infty)$. Hence by Theorem 2 we have $(T) < 1$.

Therefore the result follows from Theorem 1.

Q.E.D.

Corollary 3 extends a result obtained in [1].

COROLLARY 4. *Let* $T: E \multimap E$ *be an upper semicontinous acyclic-valued map. Assume that* T *is asymptotically linear and* α*-contractive. If*

$$\lambda x \neq T'_\infty(x) \qquad \text{for all} \quad x \neq 0 \qquad \text{and} \quad \lambda \geq 1,$$

then $I - T$ *is onto.*

Proof. T'_∞ is an α-contraction with the same constant (see [10]) and $r^+(T'_\infty) < 1$ (see [4]). Hence the result is a consequence of Corollary 3.

Q.E.D.

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