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A Remark on Set-Valued Mappings that Satisfy the Leray-Schauder Condition

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Analisi funzionale. — *A Remark on Set-Valued Mappings that Satisfy the Leray-Schauder Condition.* Nota [(*)] di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia K un sottoinsieme convesso e quasi completo di uno spazio vettoriale topologico E, localmente convesso e di Hausdorff. Sia $F : K \multimap E$ un'applicazione multivoca addensante e con immagine limitata. Si dimostra (con un ragionamento elementare) che se F soddisfa la condizione di Leray-Schauder ed è semicontinua superiormente allora possiede un punto fisso.

Let K be a quasi-complete convex subset of a locally convex Hausdorff topological vector space E, and let F assign to each point in K a nonempty compact convex subset of E. Assume that K has a nonempty interior and that F is condensing and has a bounded range. It has been recently shown [10, Theorem 2] that if F satisfies the Leray-Schauder boundary condition and is continuous, then it has a fixed point. In view of the well-known Fan-Glicksberg fixed point theorem [1, p. 122 and 5, p. 171], it is natural to ask if this result remains true when F is assumed to be merely upper semicontinuous. The purpose of this note is to answer this question in the affirmative. Our method of proof is very simple. It differs from the one employed in [10]. In the sequel the convex hull, closed convex hull, interior, and boundary of a subset D of E will be denoted by *co* (D), *clco* (D), *int* (D), and *bdy* (D) respectively.

THEOREM. *Let* K *be a closed convex subset of a locally convex Hausdorff topological vector space* E, *and let an upper semicontinuous* F *assign to each point in* K *a nonempty compact convex subset of* E. *Suppose that* F *has a bounded range and that there is a point* w *in* int (K) *such that*

(L-S) *for every* $y \in bdy(K)$ *and* $z \in F(y)$,

$$z - w \neq m(y - w) \quad \text{for all} \quad m > 1 .$$

If either

(*a*) F *is condensing and* K *is quasi-complete, or*

(*b*) F (K) *is relatively compact,*
then F *has a fixed point.*

Proof. We may and shall assume that $w = 0$. Let q be the Minkowski functional of K. Define a continuous retraction $r : E \to K$ by $r(x) = x$ for $x \in K$ and $r(x) = x/q(x)$ for $x \notin K$. Let G be the composition $F \circ r$. Note that G is also upper semicontinuous. If (*a*) holds, let $A = \{D \subset K : 0 \in D, D$ is closed and convex, and $r(clco(F(D))) \subset D\}$ and $C = \bigcap \{D : D \in A\}$.

(*) Pervenuta all'Accademia il 14 settembre 1976.

Let $B = clco\,(r\,(clco\,(F\,(C))) \cup \{0\})$. We have $B = C$ because both B and C belong to A. Since F is condensing and $r\,(D) \subset co\,(D \cup \{0\})$ for all $D \subset E$, it follows that C is totally bounded. In fact, C is compact because it is a closed subset of K and K is quasi-complete. F (C) is also compact because the image of a compact set under an upper semicontinuous mapping with compact point images is compact. Thus the image of *clco* (F (C)) under G is contained in the compact subset F (C) of *clco* (F (C)). If (*b*) holds, then the image of E under G is contained in F (K) which is also contained in a compact subset of E. By [6, Theorem 2] G has a fixed point x in both cases. If $x \notin K$, then $r\,(x) \in bdy\,(K)$, $q\,(x) > 1$, and $q\,(x)\,r\,(x) \in F\,(r\,(x))$. This contradicts (L–S). Hence $x \in K$ is a fixed point of F.

This result partially substantiates [9, Conjecture 4.4]. It also includes [7, Theorem 2] (where E is Banach and Vietoris' mapping theorem is used) and [3, Theorem 1.1] (where E is Fréchet and approximate selections are used). In fact, it can be shown (with essentially the same proof) that the Theorem remains true when F assigns to each point in K a compact acyclic subset of a metrizable E. In this way we obtain extensions of [2, Theorem 2], [4, Theorem 8], and [8, Theorem 1].

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