Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Marco Biroli

A result on convergence for almost periodic solutions of some parabolic variational inequalities

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.3-4, p. 183–192.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_3-4_183_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Analisi matematica. — *A result on convergence for almost periodic solutions of some parabolic variational inequalities.* Nota (*) di MARCO BIROLI (**), presentata dal Corrisp. L. AMERIO.

RIASSUNTO. — Si dimostra un risultato di convergenza per le soluzioni quasi periodiche di certe disequazioni variazionali paraboliche e si da una applicazione di tale risultato al « problema di omegeneizzazione ».

§ 1. INTRODUCTION AND RESULTS

We give in this work a result on convergence of almost periodic solutions of some variational inequalities and we apply this result to " homogeneisation problem ", [2], for these solutions.

Let be V an uniformly convex Banach space with norm $\| \ \|$, V^* the dual of V, which we suppose uniformly convex, $\langle \ , \ \rangle$ the duality between V and V^* and $\| \ \|_*$ the dual norm on V^*.

Let be H a separable Hilbert space with scalar product $(,)$ and norm $| \ |$.

We suppose that $V \hookrightarrow H$ is densely and compactly embedded in H.

Let be $A^\varepsilon(t) : V \to V^*$, $A(t) : V \to V^*$, $t \in R$ $\varepsilon > 0$, a family of bounded, hemicontinuous monotone operators such that

(1,1) $t \to A^\varepsilon(t)\, v \quad , \quad t \to A(t)\, v$ are measurable in V^*, $\forall v \in v$

(1,2) $\langle A^\varepsilon(t)\, v , v \rangle \geq \alpha \| v \|^p \quad \forall v \in V ; \alpha > 0 \quad , \quad p > 1$, a.e. on R

$\langle A\ (t)\, v , v \rangle \geq \alpha \| v \|^p \quad \forall v \in V ; \alpha > 0 \quad , \quad p > 1$, a.e. on R

(1,3) $\langle A^\varepsilon(t)\, v - A^\varepsilon(t)\, w , v - w \rangle \geq \alpha \| v - w \|^p \quad \forall v , w \in V$; a.e. on R

$\langle A\ (t)\, v - A\ (t)\, w , v - w \rangle \geq \alpha \| v - w \|^p \quad \forall v , w \in V$; a.e. on R

(1,4) $\| A^\varepsilon(t)\, v \|_* \leq \beta \| v \|^{p-1} + d(t) \quad \forall v \in V$; a.e. on R

$\| A\ (t)\, v \| \leq \beta \| v \|^{p-1} + d(t) \quad \forall v \in V$; a.e. on R

where $d(t + \eta) \in \mathscr{L}^\infty(R , \mathscr{L}_n^{p'}(0 , 1))$.

(1,5) $$t \to A^\varepsilon(t + \eta)\, v(\eta) \quad , \quad t \to A(t + \eta)\, v(\eta)$$

are almost periodic in $\mathscr{L}^{p'}(0 , 1 ; V^*)$ $\forall v(\eta) \in \mathscr{L}^p(0 , 1 ; V)$.

(*) Pervenuta all'Accademia il 31 ottobre 1976.
(**) Istituto di Matematica del Politecnico di Milano.

Consider, for $g(t)$ in $\mathscr{L}^{p'}([t_0, +\infty[, V^*)$, the problems

$$(1,6_\varepsilon) \qquad \frac{dv_\varepsilon}{dt}(t) + A^\varepsilon(t)\, v_\varepsilon(t) = g(t)$$

$$v_\varepsilon(t_0) = v_0$$

$$(1,6) \qquad \frac{dv}{dt}(t) + A(t)\, v(t) = g(t)$$

$$v(t_0) = v_0 .$$

It is well known that problem $(1,6_\varepsilon)$, $((1,6))$ has a unique solution $v_\varepsilon(t)$ $(v(t))$, [8].

We suppose

$$(1,7) \qquad \lim_{\varepsilon\to 0}{}^{*}\, v^\varepsilon(t) = v(t) \qquad \textit{in} \quad \mathscr{L}^p([t_0, +\infty[; V) .$$

Let $K \subset H$ be a closed convex set such that

$$(1,8) \qquad (I + \lambda A^\varepsilon(t))^{-1} K \subset K , \qquad \lambda > 0$$

a.e. on R and $f(t) \in S^{p'} — AP(R; V^*)$ [$AP(R; E)$ = space of E-almost periodic functions, $S^q — AP(R; E)$ = space of $E — S^q$ almost periodic functions; E is a Banach space].

We consider the problems.

$$(1,9_\varepsilon) \qquad \int_{t_1}^{t_2} \left\langle \frac{dv}{dt}(t) + A^\varepsilon(t)\, u_\varepsilon(t) - f(t), v(t) - u_\varepsilon(t) \right\rangle dt \geq$$

$$\geq \frac{1}{2} | v(t_2) - u_\varepsilon(t_2) |^2 - \frac{1}{2} | v(t_1) - u_\varepsilon(t_1) | ; \qquad t_2 \geq t_1$$

$$\forall v(t) \in H^1_{loc}(R; H) \cap \mathscr{L}^p_{loc}(R; V) \quad , \quad v(t) \in K$$

$$u_\varepsilon(t) \in AP(R; H) \cap S^p — AP(R; V) \quad , \quad u_\varepsilon(t) \in K$$

$$(1,9) \qquad \int_{t_1}^{t_2} \left\langle \frac{dv}{dt}(t) + A(t)\, u(t) - f(t), v(t) - u(t) \right\rangle dt \geq$$

$$\geq \frac{1}{2} | v(t_2) - u(t_2) |^2 - \frac{1}{2} | v(t_1) - u(t_1) | ; \qquad t_2 \geq t_1$$

$$\forall v(t) \in H^1_{loc}(R; H) \cap \mathscr{L}^p_{loc}(R; V) \quad , \quad v(t) \in K$$

$$u_\varepsilon(t) \in AP(R; H) \cap S^p — AP(R; V) \quad , \quad u(t) \in K .$$

We show the following result:

THEOREM 1. *Problem* (1,9) ((1,9$_\varepsilon$)) *has a unique solution* $u(t)$ ($u_\varepsilon(t)$) *and we have*

$$\lim_{\varepsilon\to 0} u_\varepsilon(t) = u(t) \quad \text{in} \quad \mathscr{L}^\infty_{\text{loc}}(\mathrm{R}\,;\mathrm{V}^*)$$

$$\lim_{\varepsilon\to 0}{}^* u_\varepsilon(t) = u(t) \quad \text{in} \quad \mathscr{L}^p_{\text{loc}}(\mathrm{R}\,;\mathrm{V}).$$

The proof of Theorem 1 is divided into three steps:

1) a preliminary result on Cauchy's problem for parabolic variational inequalities related to (1,6), (1,6$_\varepsilon$) and to the convex K;

2) we show the result for smooth $f(t)$;

3) we finally show the result in the general case by regularisation on $f(t)$.

In § 2 we give the preliminary result on Cauchy's problem, in § 3 we show Theorem 1 and in § 4 we give an application of Theorem 1 to the "homogeneisation problem".

§ 2. PRELYMINARY RESULT ON CAUCHY'S PROBLEM

We give here a result on Cauchy's problem for parabolic variational inequalities, which is interesting indipendently of the proof of Theorem 1.
Let be $g_\varepsilon(t) \in \mathscr{L}^{p'}_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}^*)$, $u_{0\varepsilon}\,, u_0 \in \mathrm{K}$ with

$$\lim_{\varepsilon\to 0} g_\varepsilon(t) = g(t) \quad \text{in} \quad \mathscr{L}^{p'}_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}^*)$$

$$\lim_{\varepsilon\to 0} u_{0\varepsilon} = u_0 \quad \text{in} \quad \mathrm{H}\,.$$

We consider the problems

$$(2,1_\varepsilon) \qquad \int_0^s \left\langle \frac{dv}{dt}(t) + \mathrm{A}_\varepsilon(t)\,u_\varepsilon(t) + g_\varepsilon(t)\,, v(t) - u_\varepsilon(t)\right\rangle dt \geq$$
$$\geq \frac{1}{2}\,|v(s) - u_\varepsilon(s)|^2 - \frac{1}{2}\,|v(0) - u_{0\varepsilon}|^2 \quad , \quad s \geq 0\,,$$

$$\forall v(t) \in \mathrm{H}^1_{\text{loc}}(\mathrm{R}_+\,;\mathrm{H}) \cap \mathscr{L}^p_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}) \quad , \quad v(t) \in \mathrm{K}$$

$$u_\varepsilon(t) \in \mathscr{L}^p_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}) \cap \mathrm{C}(\mathrm{R}_+\,;\mathrm{H}) \quad , \quad u_\varepsilon(t)\,(t) \in \mathrm{K} \quad , \quad u_\varepsilon(0) = u_{0\varepsilon}$$

$$(2,1) \qquad \int_0^s \left\langle \frac{dv}{dt} + \mathrm{A}(t)\,u(t) - g(t)\,, v(t) - u(t)\right\rangle dt \geq$$
$$\geq \frac{1}{2}\,|v(s) - u(s)|^2 - \frac{1}{2}\,|v(0) - u_0|^2 \quad , \quad s \geq 0\,.$$

$$\forall v(t) \in \mathrm{H}^1_{\text{loc}}(\mathrm{R}_+\,;\mathrm{H}) \cap \mathscr{L}^p_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}) \quad , \quad v(t) \in \mathrm{K}$$

$$u(t) \in \mathscr{L}^p_{\text{loc}}(\mathrm{R}_+\,;\mathrm{V}) \cap \mathrm{C}(\mathrm{R}_+\,;\mathrm{H}) \quad , \quad u(t) \in \mathrm{K} \quad , \quad u(0) = u_0\,.$$

We suppose that the hypotheses on $A^{\varepsilon}(t), A(t)$, given in the § 1, valid; we have.

THEOREM 2. *Let be* $u(t)\,(u_{\varepsilon}(t))$ *the solution of* (2,1) ((2, 1$_{\varepsilon}$)); *we have*

$$\lim_{\varepsilon\to 0} u_{\varepsilon}(t) = u(t) \quad \text{in} \quad \mathcal{L}^{\infty}_{\text{loc}}(R_+; V^*)$$

$$\lim_{\varepsilon\to 0}{}^{*} u_{\varepsilon}(t) = u(t) \quad \text{in} \quad \mathcal{L}^{p}_{\text{loc}}(R_+; V).$$

From well—known continuous—dependence properties the result is showed if we show this result in the case $g_{\varepsilon}(t) = g(t)$, $u_{0\varepsilon} = u_0$, $g(t) \in \mathcal{L}^{2}_{\text{loc}}(R_+; H)$.

Consider now the penalised problems

$$(2,2_{\varepsilon}) \qquad \frac{du_{\varepsilon\lambda}}{dt}(t) + A^{\varepsilon}(t)\,u_{\varepsilon,\lambda}(t) + \frac{1}{\lambda}(u_{\varepsilon,\lambda}(t) - P_k\,u_{\varepsilon,\lambda}(t)) = g(t)$$

$$u_{\varepsilon,\lambda}(0) = u_0$$

$$(2,2) \qquad \frac{du_{\lambda}}{dt}(t) + A(t) + \frac{1}{\lambda}(u_{\lambda}(t) - P_k\,u_{\lambda}(t)) = g(t)$$

$$u(0) = u_0$$

where P_k is the projection on K in H.

By (2,2$_{\varepsilon}$), multipling for $\frac{1}{\lambda}(u_{\varepsilon,\lambda}(t) - P_k\,u_{\varepsilon,\lambda}(t))$ and from [6] we have

$$(2,3) \qquad \int_0^T \frac{1}{\lambda^2}\,|u_{\varepsilon,\lambda}(t) - P_k\,u_{\varepsilon,\lambda}(t)|^2\,dt \le C_T,$$

(C_T indicates a constant dependent on T).

From (2,2$_{\varepsilon}$), multipling by $u_{\varepsilon}(t)$, we have

$$(2,4) \qquad \int_0^T \|u_{\varepsilon,\lambda}(t)\|^p\,dt \le C_T.$$

From (2,2$_{\varepsilon}$) (2,3) (2,4) we have

$$(2,5) \qquad \left\|\frac{du_{\varepsilon,\lambda}}{dt}(t)\right\|_{\mathcal{L}^{p'}(0,T;V^*)+\mathcal{L}^{2}(0,T;H)} \le C_T.$$

From (2,4) (2,5) we have, at least for a subsequence,

$$(2,6) \qquad \lim_{\varepsilon\to 0} u_{\varepsilon,\lambda}(t) = w(t) \quad \text{in} \quad \mathcal{L}^{\infty}_{\text{loc}}(R_+; V^*)$$

$$\lim_{\varepsilon\to 0}{}^{*} u_{\varepsilon,\lambda}(t) = w(t) \quad \text{in} \quad \mathcal{L}^{p}_{\text{loc}}(R_+; V)$$

$$\lim_{\varepsilon\to 0} u_{\varepsilon,\lambda}(t) = w(t) \quad \text{in} \quad \mathcal{L}^{2}_{\text{loc}}(R_+; H).$$

Being, from (2,6),

$$\lim_{\varepsilon\to 0} (u_{\varepsilon,\lambda}(t) - P_k u_{\varepsilon,\lambda}(t)) = (w(t) - P_k w(t)) \tag{2,7}$$

in $\mathscr{L}^2_{loc}(R_+ ; H)$.

From (1,7) (2,7) and from some well known continuous dependence results we have $w(t) = u_\lambda(t)$ and (2,6) for the sequence $u_{\varepsilon,\lambda}(t)$.

We observe now that in (2,3) C_T is independent of λ, then by the same methods used in [6], p. 57, we have

$$\lim_{\lambda\to 0} u_{\varepsilon,\lambda}(t) = u_\varepsilon(t) \quad \text{in} \quad \mathscr{L}^p_{loc}(R_+ ; V) \cap \mathscr{L}^\infty_{loc}(R_+ ; H) \tag{2,8}$$

$$\lim_{\lambda\to 0} u_\lambda(t) = u(t) \quad \text{in} \quad \mathscr{L}^p_{loc}(R_+ ; V) \cap \mathscr{L}^\infty_{loc}(R_+ ; H)$$

uniformly for ε.

Fom (2,6) and (2,8) we have the result.

Remark 1. The idea used in the proof of Theorem 2 derives from an idea used by J. L. Lions in [3] for the elliptic case.

§ 3. PROOF OF THEOREM 1

LEMMA 1. *The problem* (1,9) ((1,9$_\varepsilon$)) *has a unique solution.*

The proof of this lemma is an easy modification of the proof of Theorem 12 in [5].

LEMMA 2. *Let be* $f \in AP(R ; V^*)$; *there is a sequence* $f_n \in AP(R ; H)$ *such that*

$$\lim_{n\to\infty} f_n = f \quad \text{in} \quad \mathscr{L}^\infty(R ; V^*) .$$

Let be $\{v_n\}_{n=1}^\infty$ an orthonormal basis in H; $\{v_n\}_{n=1}^\infty$ is also a basis in V*.

Let V_n be the space spanned by $\{v_n\}_{n=1}^m$ and P_{V_n} the projection on V_n in V*.

V* being an uniformly convex Banach space, P_{V_n} is a continuous operator. Since $f \in AP(R ; V^*)$, its trajectory has a compact closure in V*.

Let $f_n(t) = P_{V_n} f(t)$.

From the continuity of P_{V_n} we have $f_n \in AP(R ; V^*)$ and then $f_n \in AP(R ; H)$.

Let $\eta > 0$; we can choose r points w_i, $i = 1, \cdots, r$, such that the trajectory of f is contained in the set $\bigcup_{i=1}^r B\left(w_i, \frac{\eta}{3}\right)$ $\left[B\left(w_i ; \frac{\eta}{3}\right) = \text{ball with radius } \frac{\eta}{3} \text{ and center } w_i\right]$.

Let $\bar{n}_\eta$ be such that

$$\|w_i - P_{V_n} w_i\|_* \leq \frac{\eta}{3} \qquad i = 1, \cdots, 2 \qquad n \geq \bar{n}_\eta ,$$

we have easily

$$\|f_n(t) - f(t)\|_* \leq \eta$$

then

$$\lim_{n\to\infty} f_n(t) = f(t) \qquad \text{in} \quad \mathscr{L}^\infty(\mathrm{R}\,;\mathrm{V}^*)$$

COROLLARY 1. *Let* $f \in \mathrm{S}^{p'} - \mathrm{AP}(\mathrm{R}\,;\mathrm{V}^*)$; *there is a sequence* $f_n \in \mathrm{S}^2 - \mathrm{AP}(\mathrm{R}\,;\mathrm{H})$ *such that*

$$\lim_{n\to\infty} f_n(t+\eta) = f(t+\eta) \qquad \text{in} \quad \mathscr{L}^{p'}(0\,,1\,;\mathrm{V}^*)$$

uniformly in t.

If $p \geq 2$ the result is an immediate consequence of Lemma 1, if $p < 2$ choosing a basis in $\mathscr{L}^2(0\,,1\,;\mathrm{H})$, which is also in $\mathscr{L}^{p'}(0\,,1\,;\mathrm{H})$ and repeating the proof of the Lemma 1, we have the result.

LEMMA 3. *Let* $u_1(t)\,(u_2(t))$ *be the solution of* (1,9) *relative to* $f(t) = f_1(t)$ $(f(t) = f_2(t))$; *and suppose*

$$\|f_1(t+\eta) - f_2(t+\eta)\|_{\mathscr{L}^{p'}_\eta(0,1;\mathrm{V}^*)} \leq \delta \qquad (\delta < 0)$$

then

$$|u_1(t) - u_2(t)| + \|u_1(t+\eta) - u_2(t+\eta)\|_{\mathscr{L}^p_\eta(0,1;\mathrm{V})} \leq \mathrm{C}(\delta)$$

where

$$\lim_{\delta\to 0+} \mathrm{C}(\delta) = 0\,.$$

We have by the same methods of [4]

$$\frac{1}{2}|u_1(t) - u_2(t)|^2 - \frac{1}{2}|u_1(t-n) - u_2(t-n)|^2 \leq$$

$$\leq \int_{t-n}^{t} (\langle f_1(s) - f_2(s)\,, u_1(s) - u_2(s)\rangle - \alpha\,\|u_1(s) - u_2(s)\|^p)\,\mathrm{d}s$$

then

$$\frac{1}{2}|u_1(\bar t - n) - u_2(\bar t - n)|^2 \geq \sum_{j=1}^{n} \int_0^1 (\alpha\,\|u_1(\bar t - j + s) - u_2(\bar t - j + s)\|^p\,\mathrm{d}s$$

$$- \sum_{j=1}^{n} \mathrm{C}\delta \left(\int_0^1 \|u_1(\bar t - j + s) - u_2(\bar t - j + s)\|^p\,\mathrm{d}s\right)^{1/p}.$$

The first term being bounded, by an easy proof ab absurdo we have

$$(3,1) \qquad \int_0^1 \|u_1(\bar t - n + s) - u_2(\bar t - r + s)\|^p\,\mathrm{d}s \leq \mathrm{C}_1 < \delta\,, \qquad n \leq \bar n_1$$

where $\lim_{\delta\to\infty+} \mathrm{C}_1(\delta) = 0\,.$

Analogously we can show

$$(3,2) \qquad \int_0^1 \| u_1(\bar{t}+n+s) - u_2(\bar{t}+n+s) \|^p \, ds \leq C_2(\delta), \qquad n \leq \bar{n}_2 .$$

From (3,1) (3,2) we have

$$(3,3) \qquad | u_1(t) - u_2(t) |^2 \leq \max (C_1(\delta), C_2(\delta)) + $$
$$+ C \left(\int_{\bar{t}-\bar{n}_1}^{\bar{t}+n_2} \| f_1(s) - f_2(s) \|_*^{p'} \, ds \right) + \delta^{p'}$$
$$\leq \max (C_1(\delta)), C_2(\delta) + [C(\bar{n}_1 + \bar{n}_2) + 1] \, \delta \|^{p'}$$

and

$$(3,4) \qquad \int_0^1 \| u_1(t+\eta) - u_2(t+\eta) \|^p \, dt \leq \frac{1}{2} | u_1(t) - u_2(t) |^2 + C\delta^{p'} - C_3(\delta)$$

where

$$\lim_{\delta \to 0+} C_3(\delta) = 0 .$$

We now show Theorem 1.

From Corollary 1 and Lemma 3 we can suppose $f \in S^2 - AP(R; H)$.

From Lemma 1 problem (1,9) ((1,9$_\varepsilon$)) has a solution $u(t)$ ($u_\varepsilon(t)$).

From the proof of Theorem 2 we have

$$(3,5) \qquad \left\| \frac{du_\varepsilon}{dt}(t) \right\|_{\mathscr{L}^{p'}(-T,T;V^*) + \mathscr{L}^2(-T,T;H)} \leq C_T$$

(where C_T is a constant dependent on T)

$$(3,6) \qquad \| u_\varepsilon(t) \|_{\mathscr{L}^p(-T,T;V^*)} \leq C_T .$$

From Lemma 2 and Theorem 12 [5] we have also

$$(3,7) \qquad | u_\varepsilon(t) | \leq C .$$

From (3,5) (3,6) (3,7) we have, at least for a subsequence,

$$(3,8) \qquad \lim_{\varepsilon \to 0}{}^* \frac{du_\varepsilon}{dt}(t) = \frac{dw}{dt}(t) \quad \text{in} \quad \mathscr{L}^{p'}_{\text{loc}}(R; V^*) + \mathscr{L}^2_{\text{loc}}(R; H)$$

$$(3,9) \qquad \lim_{\varepsilon \to 0}{}^* u_\varepsilon(t) = w(t) \quad \text{in} \quad \mathscr{L}^p_{\text{loc}}(R; V)$$

$$(3,10) \qquad \lim_{\varepsilon \to 0} u_\varepsilon(t) = w(t) \quad \text{in} \quad \mathscr{L}^2_{\text{loc}}(R_+; H) .$$

From (3,8) (3,9) (3,10) and Theorem 2 we have

$$(3,11) \qquad \left\langle \frac{dw}{dt}(t) + A(t)\, w(t) - f(t), v(t) - w(t) \right\rangle \geq 0$$

$$\forall v(t) \in H^1_{loc}(R; H) \cap \mathscr{L}^p_{loc}(R; V)\ v(t) \in K$$

$$w(t) \in \mathscr{L}^p_{loc}(R; V) \cap C(R; H) \cap \mathscr{L}^\infty(R; H), w(t) \in K$$

and as in [4] we can show that $w(t)$ is the unique solution of (3,11) then $w(t) = u(t)$.

§ 4. An application of Theorem 1

Let $\Omega \subset R^N$ be a bounded open set with smooth boundary $\Gamma, Y \subset R^N$ a right parallelepiped with edges parallel to the axes.

Let $a_{ij}(y, \tau)$ be functions on $Y \times [0, \tau_0]$ with

$$a_{ij}(y, \tau) \in \mathscr{L}^\infty(Y \times [0, \tau_0]) \quad , \quad \sum_{ij=1}^{N} a_{ij}(y, \tau)\, \xi_i \xi_j \geq \alpha |\xi|^2 \quad \forall \xi \in R^N\ (\alpha > 0).$$

We indicate again by $a_{ij}(y, \tau)$ the prolongation of $a_{ij}(y, \tau)$ to R^{N+1} by periodicity.

Let

$$V = H^1_0(\Omega), H = \mathscr{L}^2(\Omega), K = \{v(x) \in \mathscr{L}^2(\Omega), v(x) \geq 0 \text{ a.e. on } \Omega\},$$

$$(4,1) \qquad \langle A^\varepsilon(t)\, u, v\rangle = \int_\Omega \sum_{ij=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx \qquad \forall u, v \in H^1_0(\Omega)$$

We indicate $W(Y) = \{\varphi \in H^1(Y), \varphi \text{ periodic}\}$ and

$$(4,2) \qquad a_Y(\tau; \varphi, \varphi) = \int_Y \sum_{ij=1}^{N} a_{ij}(y, \tau) \frac{\partial \varphi}{\partial y_j} \frac{\partial \psi}{\partial y_i}\, dy.$$

We observe that the bilinear form (4,2) is coercive on $W(Y)/R$.

We consider now the function $\chi_j(y, \tau)$, which is defined, with the indetermination of a constant, by the problem

$$(4,3) \qquad \left(\frac{\partial \chi_i}{\partial \tau}(\tau), \psi\right)_{\mathscr{L}^2(Y)} + a_Y(\tau; \chi_j \tau), \psi) = a_Y(\tau; u_j, \psi) \quad \forall \psi \in W(Y)/R.$$

$$\chi_j(0) = \chi_j(\tau_0) \qquad \chi_j(\tau) \in W(Y)/R.$$

Consider now

$$q_{ij} = \frac{1}{|Y|\, \tau_0} \int_0^{\tau_0} \left[a_Y(\tau; \chi_j(\tau) - y_j, \chi_i(\tau) - y_i) + \left(\frac{\partial \chi_j}{\partial \tau}(\tau), \chi_i(\tau)\right)_{\mathscr{L}^2(Y)}\right] d\lambda$$

and

$$(Au, v) = \int_\Omega \sum_{ij=1}^{N} q_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx .$$

From [9] the hypotheses of Theorems 1, 2 are valid and we can apply Theorems 1, 2.

We observe that problems $(1,9_\varepsilon)$ $(1,9)$ (if $f \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathrm{R}\,;\mathrm{H})$) are formally equivalent in this case to problems

$$(4,2_\varepsilon) \qquad \frac{\partial u}{\partial t}(x\,,t) - \sum_{ij=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial u}{\partial x_j}(x,t) \geq f(t,x) \quad \text{a.e. on } \mathrm{R}\times\Omega$$

$$u(x\,,t) \geq 0 \qquad \text{a.e. on} \quad \mathrm{R}\times\Omega$$

$$\left[\frac{\partial u}{\partial t}(x,t) - \sum_{ij=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \frac{\partial u}{\partial x_j}(x,t)\right) - f(x,t)\right] u(x,t) = 0 \quad \text{a.e. on } \mathrm{R}\times\Omega$$

$$u(x\,,t)\,|_{\mathrm{N}} = 0$$

$$u \in \mathrm{AP}(\mathrm{R}\,;\mathscr{L}^2(\Omega)) \cap \mathrm{S}^p - \mathrm{AP}(\mathrm{R}\,;\mathrm{H}^1_0(\Omega))$$

$$(4,2) \qquad \frac{\partial u}{\partial t}(x\,,t) - \sum_{ij=1}^{N} q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x\,,t) \geq f(x\,,t) \qquad \text{a.e. on} \quad \mathrm{R}\times\Omega$$

$$u(x\,,t) \geq 0 \qquad \text{a.e. on} \quad \mathrm{R}\times\Omega$$

$$\left[\frac{\partial u}{\partial t}(x\,,t) - \sum_{ij=1}^{N} q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) - f(x,t)\right] u(x,t) = 0 \quad \text{a.e. on } \mathrm{R}\times\Omega$$

$$u(x\,,t)\,|_{\Gamma} = 0$$

$$u \in \mathrm{AP}(\mathrm{R}\,;\mathscr{L}^2(\Omega)) \cap \mathrm{S}^p - \mathrm{AP}(\mathrm{R}\,;\mathrm{H}^1_0(\Omega)) .$$

Remark 1. *a*) Other applications of the Theorem can be given in relation to G-convergence for parabolic equations, [7], and to some convergence theorems for maximal monotone operators, [1].

b) By the same methods used here, one can also show the result of this paragraph in some nonlinear cases and in the case of replacement of t/ε^2 by t/ε in the preceding application, [3].

REFERENCES

[1] H. ATTOUCH (1976) – Thèse, 1 partie. Université Paris VI, 28-5-1976.

[2] A. BENSOUSSAN, J. L. LIONS and G. PAPANICOLAU (1975) – *Sur quelques phénomenes asymptotiques d'évolution*, « Comptes Rendus Acad. Sc. Paris », *A 281*, 317–322.

[3] A. BENSOUSSAN, J. L. LIONS and G. PAPANICOLAU (1976) – *Homogéneisation, correcteurs et problèmes nonlinéaires*, « Comptes Rendus Acad. Sc. Paris », *A 282*, 1277–1282.

[4] M. BIROLI (1970) – *Sull'unicità della soluzione limitata di una diseguaglianza variazionale d'evoluzione*, « Rend. Acc. Naz. Lincei », ser. VIII, *48* (4), 409–411.

[5] M. BIROLI (1972) – *Sur les solutions bornées ou pr:squ périodiques des équantions d'évolution multivoques sur un espace de Hilbert*, « Ric. di Mat. », *21*, 17–48.

[6] H. BRÉZIS (1962) – *Opérateurs maximaux monotones et semigroupes de contractions dans les espace de Hilbert*, « Lecture Notes », *5*, North–Holland.

[7] F. COLOMBINI and S. SPAGNOLO (1976) – *Sur la convergence de solutions d'équations paraboliques avec coeffcients qui dépenden du temps*, « Comptes Rendus Acad. Sc. Paris », *A 282*, 735–739.

[8] L. LIONS (1969) – *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Gauthier Villars.

[9] J. L. LIONS (1975–1976) – Corso Collège de France.