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Analisi matematica. — *Stochastic differential equations in Banach spaces, variational formulation* (*). Nota (**) di GIUSEPPE DA PRATO, MIMMO IANNELLI e LUCIANO TUBARO, presentata dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si danno risultati di esistenza e unicità, da un punto di vista variazionale, della soluzione per una equazione differenziale stocastica in spazi di Hilbert, in condizioni di non-lipschitzianità.

1. INTRODUCTION

Let us recall the definition of abstract Wiener space: let H_0 be a separable Hilbert space; we will denote the set of finite projections on H_0 by $\mathscr{F}$. Let X be a Banach space of which H_0 is a dense subspace.

DEFINITION 1. (H_0, X) *is an abstract Wiener space if for any* $\varepsilon > 0$ *there exists a projection* $P_\varepsilon \in \mathscr{F}$ *such that*

$$P \perp P_\varepsilon \Rightarrow \mu(\{x : |Px|_X > \varepsilon\}) < \varepsilon$$

where $P \in \mathscr{F}$ *and*

$$\mu(\{x : |Px|_X > \varepsilon\}) = (2\pi)^{-\frac{n}{2}} \int_{\{x \in P(H_0) : |Px|_X > \varepsilon\}} e^{-\frac{|x|^2_{H_0}}{2}} dx$$

where $n = \dim P(H_0)$.

THEOREM 1. *Let* (H_0, X) *an abstract Wiener space; if* B *is any Borel set in* $P(H_0)$, *where* $P \in \mathscr{F}$, *the measure*

$$\mu_t(P^{-1}(B)) = (2\pi t)^{-\frac{n}{2}} \int_{B \cap P(H_0)} e^{-\frac{|x|^2_{H_0}}{2t}} dx \qquad n = \dim P(H_1)$$

defined on all cylindrical sets (1) *in* H_0, *as an extension to cylindrical sets of* X *in such a way it is countably additive.*

Example. Let X be a Hilbert space, $H_0 = \sqrt{S}X$ where S is a strictly positive trace-operator on X.

(*) Lavoro eseguito nell'ambito del G.N.A.F.A del C.N.R.
(**) Pervenuta all'Accademia il 7 settembre 1976.

(1) A cylindrical set is any set like $P^{-1}(B)$ for some $P \in \mathscr{F}$ and some B, Borel set in $P(H_0)$.

Let V be a separable reflexive Banach space and H a Hilbert space such that $V \hookrightarrow H \hookrightarrow V'$ [2]. Let $(\Omega, \mathscr{E}, P)$ be a probability space, $H_0 \hookrightarrow V$ a Hilbert space such that (H_0, V) is an abstract Wiener space, $W(t)$ a Wiener process in (H_0, V) and $\mathscr{F}_t$ the smallest σ-algebra such that $W(s)$, $0 \leq s \leq t$, is measurable [3].

Given the mappings

$$f: V \to V'$$

$$G: V \to L^*_\lambda(H_0)$$

where $L^*_\lambda(H_0)$ is the space of all operators T of the form $T = \lambda I + T_0$, with λ fixed in $\mathbf{R}$ and T_0 variable in the space of Hilbert-Schmidt operators in H_0. Putting $L^*(H_0) = \bigcup_{\lambda \in \mathbf{R}} L^*_\lambda(H_0)$, it is easy to see that $L^*(H_0)$ is a Hilbert space [4]; our G's are particular functions from V in $L^*(H_0)$. We will study the stochastic Cauchy problem:

$$\text{(P)} \qquad \begin{cases} du(t) = f(u(t))\,dt + G(u(t))\,dW_t \\ u(0) = u_0 \in L^p_0(V) \ ^{(5)} \end{cases}$$

with $p \geq 2$. In the non-stochastic case $(G = 0)$, if $-f$ is assumed to be a monotone, hemicontinuous and coercive mapping, a solution of (P) is found by the Faedo-Galerkin approximation and the same method works in the case $G = I$ (see [1]). In the general case it is not possible to use this procedure for there is no general existence theorem for finite-dimensional stochastic equations and hence it is not possible to get approximate solutions of (P) without adding some other condition [6].

In this paper we show existence for (P) via the Yosida approximation in H putting on f and G suitable conditions that in the non-stochastic case reduce to the classical hypothesis used in a variational framework. These conditions have been also used in [8], together with some additional assumptions which allow to use the Faedo–Galerkin approximation.

(2) $X \hookrightarrow Y$, where X, Y are two B-spaces, means: $X \subset Y$ with continuous injections.

(3) More generally $\mathscr{F}_t$ is an one-parameter family of σ-algebras such that W_s is $\mathscr{F}_t$-measurable for $0 \leq s \leq t$ and, for $t \geq s$, $\mathscr{F}_s$ and $W_t - W_s$ are independent (see [4]).

(4) $L^*(H_0)$ is a Hilbert space with respect to scalar product

$$(T, S)_* = \sum_{\nu=1}^{\infty} (Tf_\nu, Sf_\nu)$$

where $\{f_\nu\}$ is any orthonormal basis in H_0; the scalar product is independent from the choice of the basis.

(5) $L^p_t(X)$ is the space of functions $L^p(\Omega, \mathscr{F}_t, X)$, where X is a Banach space.

(6) Concerning existence in the finite dimensional case, see [3].

We look for a solution of (P) with the following properties:

(1) $$u \in \mathrm{C}(0, \mathrm{T}; \mathrm{L}_t^p(\mathrm{V}))$$

(2) $$f(u) \in \mathrm{L}^q(0, \mathrm{T}; \mathrm{L}_t^q(\mathrm{V}')) \qquad \frac{1}{p} + \frac{1}{q} = 1$$

(3) $$\mathrm{G}(u) \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}_t^2(\mathrm{L}^*(\mathrm{H}_0)))$$

so that the stochastic differential in (P) makes sense.

The following lemmas will be used to get estimates, actually they precise our use of the Itô formula.

LEMMA 1. *Let* $u \in \mathrm{C}(0, \mathrm{T}; \mathrm{L}_t^2(\mathrm{H}))$ *have the stochastic differential*

$$\mathrm{d}u = a(t)\,\mathrm{d}t + \mathrm{B}(t)\,\mathrm{dW}_t$$

and suppose that $a \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}_t^2(\mathrm{H}))$, $\mathrm{B} \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{L}_t^2(\mathrm{L}^*(\mathrm{H}_0)))$; *then*:

(4) $$|u(t)|^2_{\mathrm{L}^2(\mathrm{H})} = |u_0|^2_{\mathrm{L}^2(\mathrm{H})} + \int_0^t \mathrm{E}\{2(a(s), u(s)) + \|\mathrm{B}(s)\|^2_{\mathrm{L}^*(\mathrm{H}_0)}\}\,\mathrm{d}s\,.$$

Proof. Let $\varphi_k : \mathbf{R} \to \mathbf{R}$ be a sequence such that

$$\begin{cases} \varphi_k \in \mathrm{C}^2(\mathbf{R}), & \varphi_k(0) = 0, \quad |\varphi_k(r)| \leq ck, \quad \varphi_k(r) \to r \\ |\varphi_k'(r)| \leq c, & \varphi_k'(r) \to 1, \quad |\varphi_k''(r)| \leq \dfrac{c}{k}. \end{cases}$$

Define the mapping $\Phi_k : \mathrm{H} \to \mathrm{V}$ putting:

$$\Phi_k(x) = \sum_1^k {}_i\, \varphi_k(x_i)\, e_i \qquad \forall x \in \mathrm{H}$$

where $\{e_i\} \subset \mathrm{V}$ is a basis in H and $x_i = (x, e_i)$. Then we have:

$$\begin{aligned} \mathrm{d}\,|\Phi_k(u(t))|^2 = {} & [2(\Phi_k(u(t)), \Phi_k'[u(t)]\,a(t)) + \\ & + \|\Phi_k'[u(t)]\,\mathrm{B}(t)\|^2_{\mathrm{L}^*(\mathrm{H}_0)} + \\ & + (\Phi_k(u(t)), \mathrm{TR}\{\Phi_k''[u(t)](\mathrm{B}(t)\cdot, \mathrm{B}(t)\cdot)\})]\,\mathrm{d}t + \\ & + 2(\Phi_k(u(t)), \Phi_k'[u(t)]\,\mathrm{B}(t)\,\mathrm{dW}_t)\,. \end{aligned}$$

Hence:

$$\begin{aligned} |\Phi_k(u(t))|^2_{\mathrm{L}^2(\mathrm{H})} = {} & |\Phi_k(u_0)|^2_{\mathrm{L}^2(\mathrm{H})} + \\ & + \int_0^t \mathrm{E}\{2(\Phi_k(u(s)), \Phi_k'[u(s)]\,a(s)) + \|\Phi_k'[u(s)]\,\mathrm{B}(s)\|^2_{\mathrm{L}^*(\mathrm{H}_0)} + \\ & + (\Phi_k(u(s)), \mathrm{TR}\{\Phi_k''[u(s)](\mathrm{B}(s)\cdot, \mathrm{B}(s)\cdot)\})\}\,\mathrm{d}s \end{aligned}$$

and going to the limit we get (4).

LEMMA 2. *Let us suppose that*

$$u \in L^p(0, T; L_t^p(V)) \quad , \quad u(0) = u_0 \in L_0^p(V)$$

$$a \in L^q(0, T; L_t^q(V'))$$

$$A \in L^p(0, T; L_t^p(V)) \quad \textit{where} \quad A(t) = \int_0^t a(s)\, ds$$

$$B \in L^2(0, T; L_t^2(L^*(H_0)))$$

such that

$$u(t) = u_0 + \int_0^t a(s)\, ds + \int_0^t B(s)\, dW_s$$

then

$$(4') \qquad |u(t)|_{L^2(H)}^2 = |u_0|_{L^2(H)}^2 + \int_0^t E\{2\langle a(s), u(s)\rangle + \| B(s)\|_{L^*(H_0)}^2\}\, ds .$$

Proof. We state that it is possible to find a sequence $A_n \in \mathcal{D}([0,T]; L_t^p(V))$ such that

$$A_n \to A \quad \text{in} \quad L^p(0, T; L_t^p(V))$$

$$A_n' \to a \quad \text{in} \quad L^q(0, T; L_t^q(V')) .$$

Then consider

$$u_n(t) = u_0 + A_n(t) + \int_0^t B(s)\, dW_s .$$

Clearly $u_n \to u$ in $L^p(0, T; L_t^p(V))$, hence in $L^2(0, T; L_t^2(H))$; besides we can apply Lemma 1 to (4″) to get

$$|u_n(t)|_{L^2(H)}^2 = |u_0|_{L^2(H)}^2 + 2\int_0^t E\{(A_n'(s), u_n(s))\}\, ds + \int_0^t E\{\| B(s)\|_{L^*(H_0)}^2\}\, ds$$

from which we get (4′) as $n \to \infty$. At last our statement at the beginning of the proof can be proved by adapting the proof of Theorem 2.1, Chapter I, of [7].

2. DISSIPATIVITY, COERCIVITY AND THE APPROXIMATE PROBLEM

Let us consider the following assumption:

$$(H_1) \qquad 2\langle f(x) - f(y), x - y\rangle + \| G(x) - G(y)\|_{L^*(H_0)}^2 \leq 0 \qquad \forall x, y \in V.$$

The pair (f, G) will be said decreasing if (H_1) is verified. We remark that if (H_1) is verified then the mapping $-f: V \rightarrow V'$ is monotone. The first consequence of (H_1) is the following theorem:

THEOREM 2. *Assume that* (H_1) *is verified, and let* u *and* v *be solutions of* (P) *with initial data* u_0 *and* v_0 *respectively; then the following estimate is true*:

$$| u(t) - v(t) |_{L^2_t(H)} \leq | u_0 - v_0 |_{L^2_0(H)} .$$

The proof of Theorem 2 is got by Lemma 2 and it means uniqueness of the solution of (P).

Our second assumption is

$$(H_2) \qquad 2 \langle f(x), x \rangle + \| G(x) \|^2_{L^*(H_0)} \leq - \omega \| x \|^p \qquad \forall x \in V$$

where $\omega > 0$. If (H_2) is verified then the pair (f, G) will be said to be coercive; obviously (H_2) yields coercivity for the mapping $-f$.

To complete the picture we also consider the following assumptions on f:

$$(H_3) \qquad f \text{ is hemicontinuous and } \quad \| f(x) \|_{V'} \leq k \| x \|^{p-1} .$$

In the following we suppose that (H_1), (H_2), (H_3) are verified to define the approximate problems and show convergence to a solution of problem (P).

First of all we consider the following mapping:

$$(5) \qquad \tilde{f}: \begin{cases} D_{\tilde{f}} = \{x \in V \mid f(x) \in H\} \\ \tilde{f}(x) = f(x) \qquad \forall x \in D_{\tilde{f}} . \end{cases}$$

Owing to the assumptions $\tilde{f}$ is a maximal dissipative operator in H so that we can define the Yosida operators

$$(6) \qquad J_n = \left(I - \frac{1}{n} \tilde{f} \right)^{-1} : \; H \rightarrow V \qquad n > 0$$

$$(7) \qquad f_n = f \circ J_n = n (J_n - I) : \; H \rightarrow H \qquad n > 0$$

with the well known properties:

$$(8) \qquad | J_n |_L \leq 1 \quad ; \quad | f_n |_L \leq 2n .$$

Yet we define:

$$(9) \qquad G_n = G \circ J_n : \; H \rightarrow L^*(H_0) \qquad n > 0 .$$

From (H_1) is follows that G_n is Lipschitz continuous so that the approximate problem:

$$(P_n) \qquad \begin{cases} du_n(t) = f_n(u_n(t)) \, dt + G_n(u_n(t)) \, dW_t \\ u_n(0) = u_0 \end{cases}$$

has a unique solution $u_n \in C(o, T; L_t^2(H))$. In the next section we state some estimates on the sequence of approximate solutions, hence existence of one solution of (P).

3. Existence for (P)

Let $\{u_n\}$ be the sequence of solutions of (P_n). We have first:

PROPOSITION 1. *Let* (H_1), (H_2), (H_3) *be verified; then:*

(10) u_n *is a bounded sequence in* $C(o, T; L_t^2(H))$

(11) $J_n u_n$ *is a bounded sequence in* $L^p(o, T; L_t^p(V))$

(12) $f_n u_n$ *is a bounded sequence in* $L^q(o, T; L_t^q(V'))$

(13) $G_n u_n$ *is a bounded sequence in* $L^2(o, T; L_t^2(L^*(H_0)))$.

Proof. It is only worth proving (10) and (11), as (12) and (13) easily follow from these. Now from (P_n) it follows (see Lemma 1):

$$|u_n(t)|^2_{L^2(H)} = |u_0|^2_{L^2(H)} + \int_0^t E\{2\langle f_n(u_n(s)), u_n(s)\rangle \| G_n(u_n(s))\|^2_{L^*(H_0)}\} ds. \tag{14}$$

Let us remember that

$$\langle f_n(u_n(s)), u_n(s)\rangle = \langle f_n(u_n(s)), J_n(u_n(s))\rangle - \frac{1}{n} |f_n(u_n(s))|^2$$

from which, because of the coercivity,

$$|u_n(t)|^2_{L^2(H)} \leq -\omega \int_0^t \| J_n(u_n(s))\|^p_{L^p(V)} ds + |u_0|^2_{L^2(H)}.$$

We finally have:

THEOREM 3. *Let* (H_1), (H_2), (H_3) *be verified; then there exists at least one solution of problem* (P).

Proof. Let us pick from the sequence $\{u_n\}$ a subsequence [7] such that

(15) $u_n \to u$ in $L^\infty(o, T; L_t^2(H))$ weak*

(16) $J_n u_n \to v$ in $L^p(o, T; L_t^p(V))$ weak

(17) $f_n u_n \to \chi$ in $L^q(o, T; L_t^q(V'))$ weak

(18) $G_n u_n \to \psi$ in $L^2(o, T; L_t^2(L^*(H_0)))$ weak.

(7) We will denote such a subsequence $\{u_n\}$ again.

Clearly (17), (16) and (15) imply that $u = v$ and that:

$$u(t) = u_0 + \int_0^t \chi(s)\, ds + \int_0^t \psi(s)\, dW_s . \tag{19}$$

We have to show that:

$$\chi(s) = f(u(s)) \quad , \quad \psi(s) = G(u(s))$$

which will be done adapting a classical method in abstract evolution equations (see [6]).

Let us consider

$$X_n = \int_0^T E\{2(f_n(u_n(s)) - f(v(s)), J_n(u_n(s)) - v(s))\}\, ds + \\ + \int_0^T E\{\| G_n(u_n(s)) - G(v(s)) \|^2_{L^*(H_0)}\}\, ds .$$

It is $X_n \leq 0$ because of the dissipativity, on the other side:

$$X_n = 2 \int_0^T E\{(f_n(u_n(s)), J_n(u_n(s)))\}\, ds + \int_0^T E\{\| G_n(u_n(s)) \|^2_*\}\, ds + \\ - 2 \int_0^T E\{(f(v(s)), J_n(u_n(s)) - v(s))\}\, ds + \\ - \int_0^T E\{(G(v(s)), G_n(u_n(s)) - G(v(s)))_*\}\, ds + \\ - 2 \int_0^T E\{(f_n(u_n(s)), u(s)) + \frac{1}{2}(G_n(u_n(s)), G(v(s)))_*\}\, ds$$

(14) implies:

$$| u_n(T) |^2_{L^2(H)} - | u_0 |_{L^2(H)} \leq 2 \int_0^T E\{(f_n(u_n(s), J_n(u_n(s)))\}\, ds + \\ + \int_0^T E\{\| G_n(u_n(s)) \|^2_*\}\, ds$$

hence as $n \to \infty$

$$(20)\qquad \underline{\lim}\, X_n \geq |u(T)|^2_{L^2(H)} - |u_0|^2_{L^2(H)} + \\ -2\int_0^T E\{(f(v(s)), u(s)-v(s))\}\, ds + \\ -\int_0^T E\{(G(v(s)), \psi(s) - G(v(s)))_*\}\, ds + \\ -2\int_0^T E\{(\chi(s), v(s))\}\, ds + \int_0^T E\{(\psi(s), G(v(s)))_*\}\, ds.$$

Now from (19) it is:

$$(21)\qquad |u(T)|^2_{L^2(H)} = |u_0|^2_{L^2(H_0)} + 2\int_0^T E\{(\chi(s), u(s))\}\, ds + \\ + \int_0^T E\{\|\psi(s)\|^2_*\}\, ds$$

so that substituting in (20):

$$(22)\qquad 2\int_0^T E\{(\chi(s) - f(v(s)), u(s) - v(s))\}\, ds + \\ + \int_0^T E\{\|\psi(s) - G(v(s))\|^2_*\}\, ds \leq \underline{\lim}\, X_n \leq 0\,.$$

This latter inequality gives:

$$\int_0^T E\{(\chi(s) - f(v(s)), u(s) - v(s))\}\, ds \leq 0$$

and the hemicontinuity yields, by a standard argument:

$$\chi = f(u)\,.$$

On the other hand, from this, putting $v = u$ in (22) it is:

$$\int_0^T E\{\|\psi(s) - G(u(s))\|^2_*\}\, ds \leq 0$$

that is $\psi = G(u)$.

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