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Quasi-inverse involution categories, inverse categories

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Algebra. — *Quasi–inverse involution categories, inverse categories* (*). Nota (**) di MARCO GRANDIS, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Vengono enunciati alcuni risultati concernenti le categorie con involuzione, e in particolare la categoria delle relazioni (o corrispondenze) di una categoria esatta.

INTRODUCTION

This Note relates some results on regular involution categories, which are a part of a study on the componibility of induced relations in homological algebra; an outline of this study is given in the Introduction of [3], to which we refer for the purposes (rather complex) that originated the present results; the proofs and developments of these ones will appear in future works.

1. DEFINITIONS. A *regular involution category* will be a category $\mathscr{H}$ provided with an involution $j: \mathscr{H} \to \mathscr{H}$ (controvariant endofunctor, identical on objects and involutory; we write $\tilde{\alpha}$ for $j(\alpha)$) wich is *regular*, i.e. $\alpha\tilde{\alpha}\alpha = \alpha$ for any morphism α; we say that $\mathscr{H}$ is an *orthodox involution category* if, moreover, the composition of idempotent endomorphisms is idempotent (i.e. if they form a subcategory, $\mathscr{H}_1$); we say that $\mathscr{H}$ is a *quasi-inverse involution category* if, moreover, its idempotents satisfy the identity $xyxzx = xyzx$. Last, $\mathscr{H}$ is an *inverse category* if any morphism $\alpha \in \mathscr{H}(A, B)$ has a unique *generalized inverse* $\alpha^{\rho} \in \mathscr{H}(B, A)$ (this means that $\alpha = \alpha\alpha^{\rho}\alpha$ and $\alpha^{\rho} = \alpha^{\rho}\alpha\alpha^{\rho}$); then $\mathscr{H}$ is provided with a unique regular involution (i.e. $\alpha \to \alpha^{\rho}$) and its idempotents commute (when composable), hence it is quasi-inverse.

These terms (which come from semigroup theory) were introduced in [3], [4] (except quasi-inverse involution categories); more generally, [3] studies orthodox categories without involution.

2. Let $\mathscr{H}$ be a regular involution category, and $\alpha \in \mathscr{H}(A, B)$; if $\mathscr{H}(A) = \mathscr{H}(A, A)$ is the semigroup of endomorphisms of A, we considered ([4]) the *transfer mappings*:

$$\alpha_{\square} : \mathscr{H}(A) \to \mathscr{H}(B) \quad , \quad \alpha_{\square}(\varphi) = \alpha\varphi\tilde{\alpha} \tag{1}$$

$$\alpha^{\square} : \mathscr{H}(B) \to \mathscr{H}(A) \quad , \quad \alpha^{\square}(\psi) = \tilde{\alpha}\psi\alpha \tag{2}$$

$$\alpha^{\square} = (\tilde{\alpha})_{\square} \tag{3}$$

which generally do not preserve product.

(*) Lavoro eseguito nell'ambito dei gruppi di ricerca matematici del C.N.R.
(**) Pervenuta all'Accademia il 19 ottobre 1976.

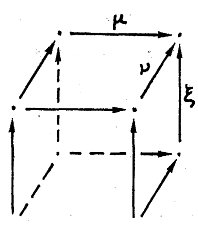
PROPOSITION. *If $\mathscr{H}$ is an orthodox involution category, the transfer mappings preserve the idempotents; moreover $\mathscr{H}$ is quasi-inverse iff for any objects A, B and any $\alpha \in \mathscr{H}(A, B)$, $\alpha_\square : \mathscr{H}_1(A) \to \mathscr{H}_1(B)$ is a morphism of semigroups ($\mathscr{H}_1(A)$ is the subsemigroup of idempotents in $\mathscr{H}(A)$).*

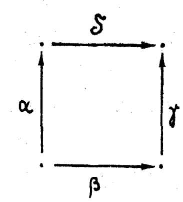
3. To any regular involution category $\mathscr{H}$ it is possible to associate an orthodox involution category $\mathscr{H}^\square$; its objects are all pairs (A, S), where A is an object of $\mathscr{H}$ and S is an involution subsemigroup of $\mathscr{H}(A)$ containing 1_A and idempotent; a morphism $(\alpha, S, T) : (A, S) \to (B, T)$ is given by an $\mathscr{H}$-morphism $\alpha : A \to B$ such that $\alpha_\square(S) \subset T$ and $\alpha^\square(T) \subset S$; the composition is obvious: $(\beta, T, U) \cdot (\alpha, S, T) = (\beta\alpha, S, U)$. $\mathscr{H}^\square$ is called the *orthodox expansion* of $\mathscr{H}$ [4].

By requiring semigroups S to be quasi-inverse, we obtain a full subcategory $\mathscr{H}^Q$ of $\mathscr{H}^\square$, which is quasi-inverse, and will be called the *quasi-inverse expansion* of $\mathscr{H}$.

4. We call *factorizing* a category having "unique" epic-monic factorizations.

THEOREM I. *A factorizing regular involution category is quasi-inverse iff any monics μ, ν, ξ having the same codomain can be imbedded in a cubic diagram of monics* (1)

(1) 

(2) 

with anticommutative faces (the square (2) is called anticommutative *if* $\beta\tilde{\alpha} = \tilde{\gamma}\delta$).

THEOREM II. *Let $\mathscr{H}$ be a factorizing involution category, $\mathscr{M}$ its subcategory of monics; these conditions are equivalent:*

a) *$\mathscr{H}$ is inverse, and its involution is regular (hence the only one to be such);*

b) *$\mathscr{M}$ has pullbacks, and any pullback square of $\mathscr{M}$ is* bicommutative *(i.e. commutative and anticommutative) in $\mathscr{H}$.*

This theorem suggests that a factorizing inverse category should be determined up to isomorphism by its subcategory of monics; in fact the *category* of factorizing inverse categories (and functors) is equivalent to the *category* of categories of monics with finite intersections (and functors preserving these ones).

5. We suppose now that $\mathscr{E}$ is an exact category, and $\mathscr{H}$ its category of relations, or correspondences (see [2], [1]); $\mathscr{H}$ is factorizing and has a regular involution.

THEOREM III. *The following are equivalent*:

a) *$\mathscr{H}$ is orthodox*;

b) *$\mathscr{H}$ is quasi-inverse*;

c) *$\mathscr{E}$ has distributive lattices of subobjects*;

c*) *$\mathscr{E}$ has distributive lattices of quotients*;

d) *direct images of subobjects, in $\mathscr{E}$, are lattice homomorphisms*;

e) *inverse images of subobjects, in $\mathscr{E}$, are lattice homomorphisms*;

e*) *direct images of quotients, in $\mathscr{E}$, are lattice homomorphisms*;

d*) *inverse images of quotients, in $\mathscr{E}$, are lattice homomorphisms.*

If these conditions are satisfied, we call $\mathscr{E}$ a *distributive exact category.*

REFERENCES

[1] H. B. BRINKMANN and D. PUPPE (1969) – *Abelsche und exacte Kategorien, Korrespondenzen*, « Lecture Notes in Mathematics », *96*, Springer, Berlin–Heidelberg–New York.

[2] M. S. CALENKO (1967) – *Correspondences over a quasi-exact category*, « Mat. Sbornik », *73*, 564–584; « Math. USSR Sbornik », *2*, 501–519.

[3] M. GRANDIS (1976) – *Canonical preorder and congruence in orthodox semigroups and categories* (*Orthodox Categories*, 1), « Boll. Un. Mat. Ital. » (5) $XIII_B$ 634–650.

[4] M. GRANDIS (1977) – *Regular and orthodox involution categories* (*Orthodox Categories*, 2), « Boll. Un. Mat. Ital. » (to appear in 1977).