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On wave solutions of coupled electromagnetic and zero-rest-mass scalar Fields in generalized Takeno space-time

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Fisica matematica. — *On wave solutions of coupled electromagnetic and zero-rest-mass scalar Fields in generalized Takeno space-time.* Nota (*) di KRISHNA B. LAL e BANSH R. MAURYA, presentata dal Socio C. CATTANEO.

RIASSUNTO. — Si studiano le equazioni di campo della relatività generale in una regione che, oltre ad un campo elettromagnetico, contenga un campo scalare associato ad un mesone aventi $\mu = c$ e si mostra che, sotto opportune condizioni, esistono delle onde soluzioni.

1. INTRODUCTION

Lal and Ali [2] have introduced the space-time metric

$$ds^2 = -A\,dx^2 - 2\,D\,dx\,dy - B\,dy^2 - \tag{1.1}$$
$$-(C - E)\,dz^2 - 2\,E\,dz\,dt + (C + E)\,dt^2,$$

where A, B, C, D are functions of the single variable $Z = Z(z - t)$ and $E = E(x, y, Z)$. The metric (1.1) reduces to Takeno's [7] general plane wave metric when $E = E(z - t)$ and it reduces to the Peres metric [4] when $A = B = C = 1$, $D = 0$ and $E = -2f(x, y, Z)$.

In the space-time metric (1.1), Lal and Ali [2] have found the solutions of the Einstein-Maxwell field equations in general relativity and other Authors ([3], [5]) have been able to find the solutions of the field equations of general relativity for regions of space containing, in addition to the electromagnetic field, a scalar-meson field V associated with a meson of rest-mass $\mu = 0$ with various space-time metrics in various symmetries. In the present paper, taking the line element (1.1), we consider the field equations of general relativity for regions of space containing the electromagnetic field and a zero-rest-mass scalar meson field. These field equation according to G. Stephenson [6] are

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2} g_{ij} R = -8\pi (E_{ij} + M_{ij}), \tag{1.2}$$

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{1.3}$$

$$F^{ij}_{;j} = 0, \tag{1.4}$$

$$g^{ij} V_{;ij} = 0, \tag{1.5}$$

(*) Pervenuta all'Accademia il 28 luglio 1976.

where R_{ij} is the Ricci tensor, F_{ij} is the antisymmetric tensor field and V is a zero-rest-mass scalar meson field. E_{ij} and M_{ij} are the electromagnetic energy-momentum tensor and energy-momentum tensor of the zero-rest-mass scalar meson field respectively, defined by

$$E_{ij} = -F_{is} F^s_j + {}^1/_4\, g_{ij} F_{sp} F^{sp}, \tag{1.6}$$

$$M_{ij} = V_{,i} V_{,j} - \tfrac{1}{2} g_{ij} V_{,s} V^{,s}. \tag{1.7}$$

Here a comma and a semicolon followed by indices denote partial and covariant differentiations respectively.

2. Calculation of G_{ij}, E_{ij} and M_{ij}

The non-vanishing components of the contravariant tensor g^{ij}, of the Cristoffel symbols of the first kind and of the Ricci tensor R_{ij} have all been found by Lal and Ali [2]. Calculating the Christoffel symbols of second kind and using the components of the various tensors mentioned above, we find

$$R = g^{ij} R_{ij} = 0 \quad \text{and} \tag{2.1}$$

$$G_{ij} = R_{ij} = P, \tag{2.2}$$

and

$$P = \left[\bar{\bar{m}} - \frac{1}{2}\, \bar{m}^2/m - \bar{m}\bar{c}/c - (\bar{A}\bar{B} - \bar{D}^2) - \right.$$
$$\left. - \left(A \frac{\partial^2 E}{\partial y^2} - 2 D \frac{\partial^2 E}{\partial x\, \partial y} + B \frac{\partial^2 E}{\partial x^2} \right) \right] \Big/ 2m .$$

A bar over a function denotes partial differentiation with respect to Z and $m = AB - D^2$.

Taking the components of the electromagnetic potentials as functions of (x, y, Z) and considering transverse electromagnetic waves propagating along the positive direction of the z-axis, Lal and Ali [2] have found that under the condition

$$\frac{\partial \rho}{\partial x} + \frac{\partial \sigma}{\partial y} = 0,$$

the components of F_{ij} are given by

$$F_{ij} = \begin{bmatrix} 0 & 0 & -\sigma & \sigma \\ 0 & 0 & \rho & -\rho \\ \sigma & -\rho & 0 & 0 \\ -\sigma & \rho & 0 & 0 \end{bmatrix} \tag{2.4}$$

The non-vanishing components of the electromagnetic energy tensor E_{ij} as obtained in [1] are

$$(2.5) \qquad E_{33} = -E_{34} = E_{44} = (A\rho^2 + 2\,D\rho\sigma + B\sigma^2)/m\,.$$

In analogy with the electromagnetic field, we take the scalar meson field V to be a function of $(x\,,y\,,Z)$. Under this assumption the components of M_{ij} are obtained as follows:

$$(2.6) \qquad \begin{aligned} M_{11} &= V_1^2 - A\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,,\\ M_{22} &= V_2^2 - B\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,,\\ M_{12} &= V_1\,V_2 - D\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,,\\ M_{13} &= -M_{14} = V_1\,\overline{V},\\ M_{23} &= -M_{24} = V_2\,\overline{V},\\ M_{33} &= \overline{V}^2 - (C - E)\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,,\\ M_{34} &= -\overline{V}^2 - E\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,,\\ M_{44} &= -\overline{V}^2 + (C + E)\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m\,, \end{aligned}$$

where suffixes 1 and 2 attached with V denote its partial derivatives with respect to x and y.

3. Solution of the field equations

The field equations (1.3) and (1.4) have been solved by Lal and Ali [2] and they have found that the field equation (1.3) is satisfied identically by the components of F_{ij} given by (2.4) with (2.3) while equation (1.4) is satisfied if

$$(3.1) \qquad \left(A\,\frac{\partial\rho}{\partial y} - B\,\frac{\partial\sigma}{\partial x}\right) - D\left(\frac{\partial\rho}{\partial x} - \frac{\partial\sigma}{\partial y}\right) = 0\,.$$

We can consider the equations (2.3) and (3.1) as simultaneous partial differential equations since A , B and D are independent of x and y and the solution of these will give ρ and σ as functions of $x\,,y$ and Z.

Taking the values of G_{ij} from (2.2), E_{ij} from (2.5) and M_{ij} from (2.6) and substituting in equation (1.2), we get

$$(3.2) \qquad V_1^2 - A\,(V_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m = 0\,,$$

$$(3.3) \qquad V_2^2 - B\,(V_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m = 0\,,$$

$$(3.4) \qquad V_1\,V_2 - D\,(V_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m = 0\,,$$

$$\mathrm{V_1\,\overline{V} = -\,V_1\,\overline{V} = 0}\,, \tag{3.5}$$

$$\mathrm{V_2\,\overline{V} = -\,V_2\,\overline{V} = 0}\,, \tag{3.6}$$

$$\begin{aligned} &-8\,\pi\,\{\overline{V}^2 - (C - E)\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m + \\ &\quad + (A\rho^2 + 2\,D\rho\sigma + B\sigma^2)/m\} = P\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &-8\,\pi\,\{\overline{V}^2 + E\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m + \\ &\quad + (A\rho^2 + 2\,D\rho\sigma + B\sigma^2)/m\} = P\,, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &-8\,\pi\,\{\overline{V}^2 + (C + E)\,(BV_1^2 - 2\,DV_1\,V_2 + AV_2^2)/2\,m + \\ &\quad + (A\rho^2 + 2\,D\rho\sigma + B\sigma^2)/m\} = P\,. \end{aligned} \tag{3.9}$$

The equations (3.5) and (3.6) are satisfied when either

(i) $\overline{V} = 0$

or (ii) $V_1 = V_2 = 0$.

When $\overline{V} = 0$, V becomes a function of x and y only, but this is not useful from the point of view of ' Wave solutions '. Therefore, we take $V_1 = V_2 = 0$ which reduces V to be a function of Z only. Using $V_1 = V_2 = 0$ and $V = V(Z)$ in the equations (3.2)-(3.9), we have

$$-8\,\pi\,\{\overline{V}^2 + (A\rho^2 + 2\,D\rho\sigma + B\sigma^2)/m\} = P\,. \tag{3.10}$$

It may be pointed out that equation (3.10) is only one condition for eight functions and we get infinitely many sets of $\{(A, B, C, D, V), (\rho, \sigma, E)\}$ satisfying the relation (3.10) and hence it admits solutions with great arbitrariness.

Substituting the values of g^{ij} from [2] and Christoffel symbols $\{^{\,i}_{jk}\}$ of second kind in the scalar wave equation (1.5), we have

$$B\,\frac{\partial^2 V}{\partial x^2} - 2\,D\,\frac{\partial^2 V}{\partial x\,\partial y} + A\,\frac{\partial^2 V}{\partial y^2} = 0\,. \tag{3.11}$$

Using $V_1 = V_2 = 0$ and $V = V(Z)$ in equation (3.11), it is seen that it is identically satisfied.

Thus, the wave solutions of the Einstein-Maxwell field equations in presence of a zero-rest-mass scalar meson field are composed of the g_{ij} *given by* (1.1), F_{ij} *given by* (2.4) *with* (2.3) *and* $V = V(Z)$ *under the conditions* (3.1) *and* (3.10). It is worth noting that if we put $V = 0$ in (3.10), the resulting equation is identical with (4.1) in [2] and solutions in the space-time of Takeno and Peres are derivable by taking $E = E(Z)$ and $A = B = C = 1$, $D = 0$, $E = -2f(x, y, Z)$, respectively in the solutions obtained in this paper.

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