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On plane-symmetric solutions of a scalar-tensor theory of gravitation

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Fisica matematica. — *On plane-symmetric solutions of a scalar-tensor theory of gravitation.* Nota(*) di KISHN BEHARI LAL e MOHAMMAD QAMRULLAH KHAN, presentata dal Socio C. CATTANEO.

RIASSUNTO. — Si ottiene una soluzione statica esatta della teoria scalare–tensoriale proposta da Dunn [5] nel caso della metrica impiegata da Kompaneets [8]. Per una scelta appropriata delle costanti arbitrarie la soluzione si riduce, come caso particolare, alla soluzione della teoria di Sen e Dunn ottenuta da Singh [9]. La nostra soluzione si riduce, in una situazione simile, alla soluzione delle equazioni di Einstein nel vuoto discussa in precedenza da Taub [10]. D'altra parte uno studio comparativo del comportamento singolare delle nostre soluzioni in due casi particolari, mostra che esse ammettono tipi analoghi di singolarità. Finalmente viene studiata una soluzione non statica a simmetria piana, che in un caso particolare descrive lo spazio–tempo vuoto non statico discusso da Bera [11].

1. INTRODUCTION

Various theories of gravitation have been proposed after general relativity. A viable theory of gravitation is one which satisfies three criteria: self consistency completeness and agreement with past experiment. The scalar-tensor theory of gravitation proposed by Brans-Dicke [1] is one of the few theories which have remained viable under these tests. It is well known that the general theory of relativity differs from the classical theory of gravitation largely in the geometrization of the gravitational field. Motivated by ideas of Mach, Brans and Dicke introduced an alternative theory of gravitation, widely known as Brans-Dicke theory. This theory is not purely geometrical, however, as the scalar field is introduced in a rather ad hoc manner in the Riemannian manifold.

Several attempts have been made to cast a scalar-tensor theory of gravitation in a wider geometrical context. Brans and Dicke observed in their work the formal connection between their theory and that of Jordan [2] which uses a five dimensional manifold. Ross [3] has constructed a scalar-tensor theory of gravitation using the Weyl formulation of Riemannian geometry and Sen and Dunn [4] have introduced a scalar-tensor theory modeled on a modification of Riemannian geometry suggested by Lyra. However, recently Dunn [5] has introduced a geometry which differs from the usual Reimannian geometry in that its linear connections have non vanishing torsion defined in terms of a scalar function. Based on this geometry, Dunn has formulated a scalar-tensor theory of gravitation whose field equations are identical in the vacuum case to those given by Dicke [6] in an alternate presentation of the Brans-Dicke theory.

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In the region of space-time with zero charge and mass densities, the field equations of this scalar-tensor theory are

$$(1.1) \qquad R_{ij} - \frac{1}{2} g_{ij} R = w\lambda^{-2} \left(\lambda_{,i} \lambda_{,j} - \frac{1}{2} g_{ij} \lambda_{,s} \lambda^{,s} \right),$$

$$(1.2) \qquad \frac{\partial}{\partial x^s} (\lambda_{,j} \sqrt{-g}\, g^{js}) - (\lambda_{,s} \lambda_{,i}/\lambda)\, g^{si} \sqrt{-g} = 0,$$

where $w = 6k^2$ is a constant, R_{ij} is the usual Ricci tensor, R is the curvature scalar of the metric g_{ij} and λ is the scalar field. A comma denotes partial differentiation with respect to the index which follows. The formalism of the geometry used in introducing the above field equations is such that if $k = 0$ or λ is a constant, the connection of the space-time is metric preserving and torsion-free; i.e., we have the Riemannian geometry.

Dunn has obtained static spherically symmetric solutions of these field equations and it has been found that, with a proper choice of the parameter k, this theory agrees with experimental results in the three classical tests of red shift, light deflection and perihilian advance.

In an earlier paper the present Authors [7] obtained cylindrical wave solutions of the scalar-tensor theory proposed by Dunn. The solution in a special case was shown to represent a plane wave in the sense of Bondi, Pirani and Robinson. The present paper is motivated to the investigation of plane symmetric solutions of the scalar-tensor theory under discussion. With this aim we have obtained, in this paper, static solution of the field equations (1.1) and (1.2) for the metric used by Kompaneets [8] and Lal and Khan [7]. It is found that the static plane symmetric solution of Sen and Dunn's scalar-tensor theory discussed by singh [9] is a special case of our solution. On the other hand it is found that our solution also includes the static empty space-time solution, discussed by Taub [10], as a particular case. Further we have made a comparative study of the singular behaviour of our solution of the present scalar-tensor theory and that of Taub's static plane symmetric solution of vacuum field equations of general relativity. It is concluded that both solutions admit a similar type of singularity. Lastly a non static plane symmetric solution of the field equations (1.1) and (1.2) has also been investigated, which in a particular case describes a non static empty space-time discussed by Bera [11].

2. METRIC AND THE FIELD EQUATIONS

For the purpose of our present investigation we consider a space-time whose geometry is described by the metric ([7], [8])

$$(2.1) \qquad ds^2 = -A\,(dx^1)^2 - C\,(dx^2)^2 - D\,(dx^3)^2 - 2B\,dx^2\,dx^3 + A\,(dx^4)^2,$$

where A, B, C, D are functions of x^1 and x^4 (x^1, x^2, x^3 denote space coordinates whereas x^4 corresponds to time coordinate t). For this space-time the non vanishing components of the Ricci tensor R_{ij} are given by

$$(2.2)\quad \begin{aligned} R_{11} &= \alpha_{11}/2\alpha - \alpha_1^2/4\alpha^2 + (L_{11} - L_{44}) + (B_1^2 - C_1 D_1)/2\alpha - \\ &\quad - (A_1 \alpha_1 + A_4 \alpha_4)/4A\alpha , \\ R_{44} &= \alpha_{44}/2\alpha - \alpha_4^2/4\alpha^2 - (L_{11} - L_{44}) + (B_4^2 - C_4 D_4)/2\alpha - \\ &\quad - (A_1 \alpha_1 + A_4 \alpha_4)/4A\alpha , \\ R_{14} &= \alpha_{14}/4\alpha - \alpha_1 \alpha_4/4\alpha^2 + (2 B_1 B_4 - C_1 D_4 - C_4 D_1)/4\alpha - \\ &\quad - (A_1 \alpha_4 + A_4 \alpha_1)/4A\alpha , \\ R_{22} &= (2A)^{-1} (C ; \alpha , P) \quad , \quad R_{33} = (2A)^{-1} (D ; \alpha , P) , \\ R_{23} &= (2A)^{-1} (B ; \alpha , P) , \end{aligned}$$

where the notations are as follows

$$(X ; \alpha , P) = [X_{11} - X_{44} - (2\alpha)^{-1} \{X_1 \alpha_1 - X_4 \alpha_4 + 2XP\}] ,$$

$$P \equiv (B_1^2 - B_4^2 - C_1 D_1 + C_4 D_4) ,$$

and we have used

$$(2.3)\qquad A = e^{2L} \quad , \quad \alpha \equiv CD - B^2.$$

A simple calculation shows that equation (1.1) on contraction yields $R = w\lambda^{-2} (\lambda_{,i} \lambda^{,i})$. Consequently (1.1) assumes a simple form

$$(2.4)\qquad R_{ij} = h_{,i} h_{,j} ,$$

where

$$(2.5)\qquad \lambda = e^{h/\sqrt{w}} .$$

The functional dependence of the metric tensor on the space-time coordinates obviously implies that the scalar field λ, and consequently h, may be taken as a function of x^1 and x^4 . With this assumption equations (1.2) and (2.4) simplify to

$$(2.6)\qquad h_{11} - h_{44} + (\alpha_1/2\alpha)\, h_1 - (\alpha_4/2\alpha)\, h_4 = 0 ,$$

$$(2.7)\qquad R_{11} = h_1^2 \;\; , \;\; R_{44} = h_4^2 \;\; , \;\; R_{14} = h_1 h_4 \;\; , \;\; R_{22} = R_{33} = R_{23} = 0 ,$$

where $h_{,i} = h_i$, and the lower suffixes 1 and 4, here and here after, correspond to partial differentiation with respect to x^1 and x^4 respectively.

Using (2.2), multiplying R_{22} by D, R_{23} by -2 B, R_{33} by C and adding, we obtain

$$(\sqrt{\alpha})_{11} - (\sqrt{\alpha})_{44} = 0\,, \tag{2.8}$$

where α is as given in (2.3).

3. A STATIC SOLUTION

In this section we restrict ourselves to the static case. Thus assuming that the unknown functions involved in the above equations are independent of x^4 and depend on x^1 only, equations (2.6)–(2.8), on substitution of R_{ij} from (2.2), simplify to

$$\begin{gathered} h'' + (h'\,\alpha'/2\,\alpha) = 0 \quad , \quad L'' + (L'\,\alpha'/2\,\alpha) = 0\,, \\ L'' - (L'\,\alpha'/2\,\alpha) - (B'^2 - C'\,D')/2\,\alpha = h'^2, \\ C'' - C'\,\alpha'/2\,\alpha - (C/\alpha)\,Q = 0 \quad , \quad D'' - (D'\,\alpha'/2\,\alpha) - (D/\alpha)\,Q = 0\,, \\ B'' - (B'\,\alpha'/2\,\alpha) - (B/\alpha)\,Q = 0\,, \end{gathered} \tag{3.1}$$

where $Q = (B'^2 - C'\,D')$, and a dash overhead denotes ordinary differentiation with respect to x^1.

Equation (2.8) for the static case yields a solution

$$\alpha = (k_1\,x^1 + k_2)^2. \tag{3.2}$$

On the other hand, equations (3.1) by virtue of (3.2), exhibit on integration

$$\begin{aligned} & L = (k_3/k_1)\log(k_1\,x^1 + k_2) + k_4 \quad , \quad h = (k_5/k_1)\log(k_1\,x^1 + k_2) + k_6\,, \\ & C = e^k\,(k_1\,x^1 + k_2)\cos h\left\{\frac{k_7}{k_1}\log(k_1\,x^1 + k_2)\right\}, \\ & D = e^{-k}\,(k_1\,x^1 + k_2)\cos h\left\{\frac{k_7}{k_1}\log(k_1\,x^1 + k_2)\right\}, \\ & B = (k_1\,x^1 + k_2)\sin h\left\{\frac{k_7}{k_1}\log(k_1\,x^1 + k_2)\right\}, \end{aligned} \tag{3.3}$$

where k and all k_i's are constants of integration and k_1, k_3, k_5, k_7, are related as

$$k_1\,(k_1 + 4\,k_3) = -2\,k_5^2 + k_7^2\,, \qquad k_1 = \neq 0\,. \tag{3.4}$$

Thus (3.3) and (3.4) along with (2.1) characterize a static solution of the field equations of the scalar-tensor theory, i.e., (1.1) and (1.2). On analysing this solution the following interesting cases arise.

(i) If $k_6 = k_7 = 0$, (3.3) reveals that $B = 0$ and $C = e^{2k} D$, which by a suitable coordinate transformation can be reduced to $C = D$. In this case the metric (2.1) transforms into the static plane symmetric metric of Taub, and the corresponding solution is given by the metric (2.1), where

$$\text{(3.5 a)} \qquad L \equiv \tfrac{1}{2} \log A = (k_3/k_1) \log (k_1 x^1 + k_2) + k_4 \,,$$

$$C = D = (k_1 x^1 + k_2) \quad , \quad B = 0 \,,$$

and the scalar function h is given by

$$\text{(3.5 b)} \qquad h = \sqrt{-(1/2 k_1)(k_1 + 4 k_3)} \log (k_1 x^1 + k_2) \,.$$

In order that the scalar function h be real, it is evident from (3.5 b) that $k_3/k_1 < -\frac{1}{4}$. It is at the same time interesting to note that in this case (3.5 a) and (3.5 b) along with (2.1) present the static plane symmetric solution of the of the field equations of Sen and Dunn's scalar-tensor theory obtained by Singh [9].

(ii) If $k_4 = k_5 = k_6 = k_7 = 0$, in view of (3.3), (3.4) and (i), we have $k_3/k_1 = -\frac{1}{4}$, so that the corresponding solution is obtained as

$$\text{(3.6)} \qquad A = (k_1 x^1 + k_2)^{-\frac{1}{2}} \,, \quad C = D = (k_1 x^1 + k_2) \quad , \quad B = 0.$$

Thus the metric (2.1), together with (3.6) describes an empty space-time discussed by Taub [10].

An analysis of the general solution given by (3.3) and (3.4) shows that they represent logrithmically divergent gravitational fields, and as $x^1 \to \infty$, L, B, C, D and h all tend to ∞. Moreover the solution has a singularity at $x^1 = -k_2/k_1$. Also when $k_1 = 0$, the gravitational field becomes singular. However we make a comparative study of the singular behaviour of the solutions obtained above for the cases (i) and(ii). Thus for the solutions of the field equations of the scalar tensor theory expressed by (3.5 a, b), we find that the Kretschman curvature invariant has the form:

$$\text{(3.7)} \qquad S \equiv R_{ijkl} R^{ijkl} = k_1^2 e^{-4L} \frac{3/4 k_1^2 + 8 k_3^2 - 2 k_1 k_3}{(k_1 x^1 + k_2)^4} \,.$$

It immediately follows from (3.7) that there is no intrinsic singularity when $k_1 = 0$, but as $x^1 \to -k_2/k_1$, $S \to \infty$, indicating an intrinsic singularity. Hence the static plane symmetric solution of the scalar tensor theory charactherized by (2.1) and (3.5 a, b) has the only intrinsic singularity located at $x^1 = -k_2/k_1$. On the other hand, the Kretschman scalar for the Taub's empty solutions given by (2.1) and (3.6) is obtained as

$$S = \frac{7 k_1^4}{4 (k_1 x^1 + k_2)^3}, \tag{3.8}$$

which reveals that the solution has an intrinsic singularity at $x^1 = -k_2/k_1$. Thus a comparison of the expressions for S in (3.7) and (3.8) points out that the singular structure of the static plane symmetric solution of Dunn's scalar-tensor theory of gravitation has the same feature as that of the vacuum field discussed by Taub [10].

4. A NON-STATIC SOLUTION OF (1.1) AND (1.2)

Consider a Riemanian space-time described by the metric

$$ds^2 = e^{2u} (dt^2 - dr^2 - r^2 d\varphi^2 - E^2 dz^2), \tag{4.1}$$

where r, φ, z are cylindrical polar coordinates and u and E are functions of time t alone.

By a straightforward calculation we find the non vanishing components of the Einstein tensor for the metric (4.1) to have the following values:

$$G_{11} = (1/r^2) G_{22} = (\ddot{E}/E + M) \quad , \quad G_{33} = - E^2 (2 \dot{E}\dot{u}/E - M), \tag{4.2}$$
$$G_{44} = 2 (\ddot{u} - \dot{u}^2) - M \quad ,$$

where $M = 2\ddot{u} + \dot{u}^2 + 2\dot{E}\dot{u}/E$. Here and in what follows an overhead dot denotes differentiation with regard to t, r, φ, z and t correspond x^1, x^2, x^3 and x^4 respectively.

Taking the scalar field λ as a function of t only and substituting the values of G_{ij} from (4.2) and using (2.5) the field equations (1.1) and (1.2) after a little simplification are finally given by

$$\ddot{E} + 2\dot{E}\dot{u} = 0 \quad , \quad \ddot{u} + 2\dot{u}^2 + \dot{E}\dot{u}/E = 0 \quad , \quad 2\ddot{u} + \dot{u}^2 - \tfrac{1}{2}\dot{h}^2 = 0 \quad , \tag{4.3}$$
$$\ddot{h} + 2\dot{h}\dot{u} + \dot{h}\dot{E}/E = 0 .$$

The equations (4.3) on integration yield

$$E = \mu e^{(c_1/c_2)u}, \quad h = (c_3/c_2)\, u \quad , \quad e^{(2+c_1/c_2)u} = (2 + c_1/c_2)(c_2 \mu^{-1} t + c_4), \tag{4.4}$$

where μ, c_1, c_2, c_3 and c_4 are constants of integration and

$$3 c_2^2 + 2 c_1 c_2 = -\tfrac{1}{2} c_3^2 \quad , \quad c_2 = \neq 0 \quad , \quad \mu = \neq 0. \tag{4.5}$$

From (4.4) and (4.5) it is obvious that the scalar function h will be real if $(c_1/c_2) < -3/2$, $C_2 > 0$.

If $c_3 = 0$, then $h = 0$. In this particular case we have $c_1/c_2 = -3/2$ and consequently

$$E = e^{-3u/2} \quad , \quad e^{u/2} = \tfrac{1}{2}(c_2 \mu^{-1} t + c_4). \tag{4.6}$$

The metric (4.1) with E and u given by (4.6) describes a non-static empty space-time discussed by Bera [11].

Thus (4.1) and (4.4) along with (4.5) constitute a non-static solution of the field equation of the scalar-tensor theory proposed by Dunn.

Remark. It is to be remarked here that the metric (4.1) is plane-symmetric. It admits the motion $\bar{x} = x + k_1$, $\bar{y} = y + k_2$, $\bar{z} = z + k_3$ where k_1, k_3, k_3 are constants. Further it admits rotation about the z axis. Thus the group of motions is of at least four parameters.

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