Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Bernard D. Coleman, David R. Owen

Thermodynamics of Elastic-Plastic Materials

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.1-2, p. 77–81.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_1-2_77_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1976.

Meccanica. — *Thermodynamics of Elastic-Plastic Materials* (*). Nota (**) di BERNARD D. COLEMAN e DAVID R. OWEN, presentata dal Socio Straniero C. TRUESDELL.

RIASSUNTO. — Si mettono in evidenza le restrizioni imposte dalle leggi della termodinamica alle equazioni costitutive che descrivono materiali elasto–plastici perfetti in deformazioni unidimensionali. Queste restrizioni sono determinate, senza l'ausilio dei concetti di entropia o energia libera, mediante formulazioni della prima e della seconda legge della termodinamica che involvono solo le seguenti quantità: temperatura, deformazione elastica, deformazione plastica, modulo di elasticità, deformazione che corrisponde al cedimento, calore latente di elasticità, calore latente di plasticità, e capacità termica.

1. PREFACE

We have recently proposed and explored an approach to thermodynamics in which the Second Law takes the form of a cyclic inequality, and the existence of entropy as a function of state is deduced, rather than assumed [1,2]. When applied to several classes of materials, e.g. elastic and viscous materials, materials with internal state variables, and materials with fading memory, our new formulation of thermodynamics yields restrictions on the constitutive relations for the stress, temperature, and heat flux agreeing with those obtained in treatments [3-6] which start with a differentiable entropy function and employ the Clausius-Duhem inequality. The new formulation has two advantages: (I) It permits one to determine the class of entropy functions compatible with given constitutive relations for stress, temperature, and heat flux. (II) In principle, it permits one to find thermodynamical restrictions on response functions which give experimentally observable quantities, without ever mentioning entropy or free energy. It is this second aspect of the new approach which we shall attempt to illustrate here through a brief discussion of the thermodynamics of a class of elastic-plastic materials.

(*) The research reported here was supported by the U.S. National Science Foundation through Grant No. MCS75–09453 A01 to Carnegie–Mellon University.

(**) Pervenuta all'Accademia il 28 luglio 1976.

(1) B. D. COLEMAN and D. R. OWEN «Arch. Rational Mech. Anal.», *54*, 1–104 (1974); *59*, 25–51 (1975).

(2) B. D. COLEMAN and D. R. OWEN, «Annali Mat. Pura Appl.» (IV), *59*, in press (1976).

(3) B. D. COLEMAN and W. NOLL, «Arch. Rational Mech. Anal.», *13*, 167–178 (1963).

(4) B. D. COLEMAN and V. J. MIZEL, «J. Chem. Phys.», *40*, 1116–1125 (1964).

(5) B. D. COLEMAN, «Arch. Rational Mech. Anal.», *17*, 1–46, 230–254 (1964).

(6) B. D. COLEMAN and M. E. GURTIN, «J. Chem. Phys.», *47*, 597–613 (1967).

2. Statement of the Problem

A unidimensional *perfect elastic-plastic element* is specified when five real-valued material functions are given; two of these, the *modulus* μ and the (absolute) *yield strain* α are positive functions of just the temperature, while three, the *latent elastic heat* Λ_e, *the latent plastic heat* Λ_p, and the *heat capacity* k_θ are initially taken to be functions of the elastic strain, the plastic strain, and the temperature. We assume here that the temperature θ can vary over an open interval $]\Theta_1, \Theta_2[$ with $0 \leq \Theta_1 < \Theta_2 \leq \infty$, that the elastic strain λ_e is a real number of magnitude not exceeding the absolute yield strain $\alpha(\theta)$, and that the plastic strain λ_p is a real number. The elements $(\lambda_e, \lambda_p, \theta)$ of the set

$$\Sigma = \{(\lambda_e, \lambda_p, \theta) \Big| \, |\lambda_e| \leq \alpha(\theta), \Theta_1 < \theta < \Theta_2, -\infty < \lambda_p < \infty\} \tag{2.1}$$

are the *states* of the specified perfect elastic-plastic element. We assume α and μ are twice continuously differentiable on $]\Theta_1, \Theta_2[$ and that Λ_e, Λ_p, and k_θ are continuously differentiable on Σ.

An oriented curve Γ lying in $\Sigma \subset R^3$ is here called an *admissible path* if it has a piecewise continuously differentiable parameterization $(\lambda_e(\cdot), \lambda_p(\cdot), \theta(\cdot)) : [0, t] \to \Sigma$ (with $t > 0$) whose domain $[0, t]$ contains a finite number of subintervals I_k (which we allow to be single points) such that for each τ in the interior of $\bigcup_k I_k$

$$\dot{\lambda}_p(\tau) = 0, \tag{2.2a}$$

whereas for each τ in the interior of the complement of $\bigcup_k I_k$

$$\lambda_e(\tau)\, \dot{\lambda}_p(\tau) > 0 \quad \text{and} \quad |\lambda_e(\tau)| = \alpha(\theta(\tau)). \tag{2.2b}$$

The points $(\lambda_e(0), \lambda_p(0), \theta(0))$ and $(\lambda_e(t), \lambda_p(t), \theta(t))$ are called, respectively, the *initial* and *final points* of Γ. If an oriented curve Γ obeying these conditions is a closed curve, it is referred to as a *closed admissible path.*

The set Σ of states is clearly arcwise connected. In addition, as the following easily proved theorem asserts, Σ is " admissibly connected ".

THEOREM 1. *For each pair of states* $(\lambda_e^{(1)}, \lambda_p^{(1)}, \theta^{(1)}), (\lambda_e^{(2)}, \lambda_p^{(2)}, \theta^{(2)})$, *there is an admissible path with initial point* $(\lambda_e^{(1)}, \lambda_p^{(1)}, \theta^{(1)})$ *and final point* $(\lambda_e^{(2)}, \lambda_p^{(2)}, \theta^{(2)})$.

If one notes that for a perfect elastic-plastic element the rate of working is $\mu\lambda_e(\dot{\lambda}_e + \dot{\lambda}_p)$, and the rate of addition of heat is $\Lambda_e \dot{\lambda}_e + \Lambda_p \dot{\lambda}_p + k_\theta \dot{\theta}$, then it becomes clear that the laws of thermodynamics require that the following two assertions be true [7]:

1) *For each closed admissible path* Γ,

$$\oint_{\Gamma} \Big([\Lambda_e(\lambda_e, \lambda_p, \theta) + \mu(\theta)\lambda_e]\, d\lambda_e + [\Lambda_p(\lambda_e, \lambda_p, \theta) + \mu(\theta)\lambda_e]\, d\lambda_p + k_\theta(\lambda_e, \lambda_p, \theta)\, d\theta \Big) = 0. \tag{2.3}$$

2) *For each closed admissible path* Γ,

$$\oint_{\Gamma} \left(\frac{\Lambda_e(\lambda_e, \lambda_p, \theta)}{\theta}\, d\lambda_e + \frac{\Lambda_p(\lambda_e, \lambda_p, \theta)}{\theta}\, d\lambda_p + \frac{k_\theta(\lambda_e, \lambda_p, \theta)}{\theta}\, d\theta \right) \leq 0. \tag{2.4}$$

It is convenient to introduce the material functions k_e and k_p defined by

$$k_e(\lambda_e, \lambda_p, \theta) = \Lambda_e(\lambda_e, \lambda_p, \theta) + \mu(\theta)\lambda_e, \tag{2.5}$$

and

$$k_p(\lambda_e, \lambda_p, \theta) = \Lambda_p(\lambda_e, \lambda_p, \theta) + \mu(\theta)\lambda_e. \tag{2.6}$$

In terms of k_e, k_p, and k_θ (each of which maps Σ into R), equation (2.3) can be written:

$$\oint_{\Gamma} \Big(k_e(\lambda_e, \lambda_p, \theta)\, d\lambda_e + k_p(\lambda_e, \lambda_p, \theta)\, d\lambda_p + k_\theta(\lambda_e, \lambda_p, \theta)\, d\theta \Big) = 0. \tag{2.7}$$

Our problem is that of finding the restrictions which 1) and 2) place on α, μ, k_θ, Λ_e, and Λ_p, or, α, μ, k_θ, k_e and k_p.

3. PRINCIPAL RESULTS

THEOREM 2. *In order that the assertions* 1) *and* 2) *both hold, it is necessary and sufficient that the functions* α, μ, k_θ, k_e, *and* k_p *obey items* (i), (ii), *and* (iii) *below.*

(i) *For each state* $(\lambda_e, \lambda_p, \theta)$,

$$k_\theta(\lambda_e, \lambda_p, \theta) + \tfrac{1}{2}\lambda_e^2\,\theta\mu''(\theta) = k_\theta(\alpha(\theta), \lambda_p, \theta) + \tfrac{1}{2}\alpha(\theta)^2\,\theta\mu''(\theta), \tag{3.1}$$

and

$$k_e(\lambda_e, \lambda_p, \theta) = [\mu(\theta) - \theta\mu'(\theta)]\lambda_e, \quad \text{i.e.}\ \Lambda_e(\lambda_e, \lambda_p, \theta) = -\theta\lambda_e\mu'(\theta). \tag{3.2}$$

(7) In a paper to appear in the Archive for Rational Mechanics and Analysis, we shall show how the present theory fits into the general framework introduced in reference 1 for thermodynamical systems, and we shall give the proof of Theorem 2.

(ii) *For each pair* (λ_p, θ) *in* $R \times]\Theta_1, \Theta_2[$,

$$k_p(-\alpha(\theta), \lambda_p, \theta) = k_p(\alpha(\theta), \lambda_p, \theta) \tag{3.3}$$

and

$$\frac{\partial}{\partial\theta} k_p(\alpha(\theta), \lambda_p, \theta) = \frac{\partial}{\partial\lambda_p} k_\theta(\alpha(\theta), \lambda_p, \theta) . \tag{3.4}$$ [8]

(iii) *For each polygon* Ω *lying in* $R \times]\Theta_1, \Theta_2[$,

$$\left| \iint_\Omega \theta^{-2} k_p(\alpha(\theta), \lambda_p, \theta)\, d\lambda_p\, d\theta \right| \leq \int_{\partial\Omega} \theta^{-1} \mu(\theta)\, \alpha(\theta)\, | d\lambda_p | , \tag{3.5}$$

where $\partial\Omega$ *is the closed polygonal curve bounding* Ω.

Item (i) of Theorem 2 concerns the elastic behavior of the perfect elastic-plastic material and is easily derived from 1) and 2) by arguments employing closed curves Γ in Σ which are " elastic paths " in the sense that (2.2a) holds for all τ in $[0, t]$, i.e. $\bigcup_k I_k = [0, t]$. The conclusions (ii) and (iii), on the other hand, appear to be new; their derivation from 1) and 2) employs closed admissible paths Γ on which " yielding occurs " in the sense that the interior of the set $([0, t] - \bigcup_k I_k)$ is not empty, and hence there is an open interval on which (2.2b) holds.

In view of (2.6), equaton (3.4) tells us that the latent plastic heat Λ_p is related as follows to the heat capacity k_θ and the " yield stress " $\mu(\theta)\,\alpha(\theta)$:

$$\frac{\partial}{\partial\theta} \Lambda_p(\alpha(\theta), \lambda_p, \theta) = \frac{\partial}{\partial\lambda_p} k_\theta(\alpha(\theta), \lambda_p, \theta) - \frac{d}{d\theta} [\mu(\theta)\, \alpha(\theta)] . \tag{3.6}$$

By choosing for Ω in (3.5) appropriate rectangles of narrow width, one can easily show that (3.5) yields.

$$\left| \int_{\theta^{(1)}}^{\theta^{(2)}} \theta^{-2} k_p(\alpha(\theta), \lambda_p, \theta)\, d\theta \right| \leq \frac{\mu(\theta^{(1)})\, \alpha(\theta^{(1)})}{\theta^{(1)}} + \frac{\mu(\theta^{(2)})\, \alpha(\theta^{(2)})}{\theta^{(2)}} \tag{3.7}$$

for each value of the plastic strain λ_p and each pair of temperatures $\theta^{(1)}, \theta^{(2)}$ in $]\Theta_1, \Theta_2[$.

(8) The derivatives appearing on the left in (3.4) and (3.6) are computed with only the second variable of k_p and Λ_p held fixed, e.g., $\frac{\partial}{\partial\theta} k_p(\alpha(\theta), \lambda_p, \theta) = \frac{d}{d\theta} f_{\lambda_p}(\theta)$, where, for each value of λ_p, $f_{\lambda_p}(\theta) = k_p(\alpha(\theta), \lambda_p, \theta)$.

In the special case in which there is a temperature inteval $]\theta^{(1)}, \theta^{(2)}[$ on which the yield stress $\mu(\theta)\,\alpha(\theta)$ is bounded above by a number $\tilde{S}$, and, furthermore, k_p is given by a function $\hat{k}$ of λ_p alone, the relation (3.7) yields

$$\frac{|\hat{k}(\lambda_p)|}{\tilde{S}} \leq \frac{\theta^{(2)} + \theta^{(1)}}{\theta^{(2)} - \theta^{(1)}} \tag{3.8}$$

for each value of λ_p. [9]

(9) From the experiments of W. S. FARREN and G. I. TAYLOR, « Proc. Roy. Soc. (London) », A *107*, 422–451 (1925), one may infer that for several metals the ratio $k_p(\alpha(\theta), \lambda_p, \theta)/\alpha(\theta)\,\mu(\theta)$ is approximately 1/10 in ranges of θ and λ_p in which the theory of perfect elastic-plastic behavior is applicable.