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Meccanica. — *Lyapunov stability of the Euler column.* Nota (*) di MARIO COMO (**) e ANTONIO GRIMALDI (***), presentata dal Corrisp. E. GIANGRECO.

RIASSUNTO. — Dopo aver dimostrato, per le travi elastiche caricate assialmente, alcune proprietà di differenziabilità seconda del funzionale energia in un opportuno spazio delle configurazioni, viene dimostrata la validità del criterio della energia come condizione sufficiente di stabilità secondo Lyapunov.

Si stabilisce così il collegamento tra il carico critico euleriano e la definizione generale di stabilità dinamica. Il lavoro si conclude con l'analisi del legame tra la norma del disturbo iniziale e la norma del moto perturbato.

1. INTRODUCTION

In spite of the wide applications and the numerous practical results achieved in the analysis of the buckling and post-buckling behaviour of structures by the "theory of elastic stability", the real significance of the predicted critical loads, as far as the dynamical (Lyapunov) definition of stability is concerned, is not, yet very clear. The wide subject, methods and still open problems in the theory of elastic stability of continuous bodies have been analyzed in an article by Knops and Wilkes [1]; some attempts, not completely satisfactory to prove the validity of static criteria of stability for tridimensional elastic bodies, namely the energy criteria, are given in [2, 3, 4].

This paper provides the connection between the Lyapunov definition of stability and the Euler critical load N_c of an elastic column, initially straight, and compressed by a centrally applied load N. Firstly, the space of the admissible displacements of the bar, with a suitable norm, is defined and a formulation of the energy criterion is given. It is then proven that the rectilinear configurations of the bar are Lyapunov stable if $N < N_c$. The analysis is developed according to the monodimensional model of the slender bar with the assumptions of axial inextensional deformations, that is according to the classical theory of the "elastica". In order to provide a further examination of the Lyapunov stability of the rectilinear configurations of the column, a relation is also established, for $N < N_c$, between the amplitude of the initial disturbance and the maximum amplitude of the perturbed motion.

The results obtained prove, in the case of monodimensional continuous systems, the validity of the energy criteria, if suitably formulated, as sufficient conditions of Lyapunov stability, and confirm an old "conjecture" of Hellinger [5].

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2. THE SPACE OF THE CONFIGURATIONS OF THE BAR AND THE ENERGY FUNCTIONAL

Let us consider the plane bending of an elastic slender column, initially perfectly straight, and compressed by a centrally applied dead load N. According to common notations, the axial and transverse displacements will be indicated by $w(z)$ and $v(z)$ if z is the coordinate axis. The assumption of inextensibility of the bar axis gives:

$$(1 + w')^2 + v'^2 = 1 \tag{1}$$

where $(\ )' = d(\ )/dz$. The curvature of the bar axis is then

$$\rho(z) = -\frac{v''}{(1 - v'^2)^{1/2}} . \tag{2}$$

We assume that the kinematical boundary conditions of the bar are linear and homogeneous in v or v', namely that they correspond to zero displacements or zero rotations $\varphi = \sin^{-1} v'$ at the ends $z = 0$ and $z = l$.

According to these assumptions the potential energy of the column under the thrust N is

$$E(v) = \frac{1}{2} EI \int_0^l \left[\frac{v''^2}{1 - v'^2} + \frac{2\lambda}{l^2}\left(\sqrt{1 - v'^2} - 1\right)\right] dz \tag{3}$$

where EI is the bending rigidity and

$$\lambda = \frac{Nl^2}{EI} . \tag{4}$$

Let us define the Hilbert space of all admissible configurations of the bar as the space of all functions $v(z), z \in [0, l]$, with square integrable second derivatives $v''(z)$ and satisfying the kinematical boundary conditions; the scalar product is defined as

$$[v_1, v_2] = EI \int_0^l v_1'' v_2'' \, dz \tag{5}$$

and the corresponding norm

$$|||v||| = EI \left(\int_0^l v''^2 \, dz\right)^{1/2} . \tag{6}$$

This space is commonly defined as an "energy" Hilbert space H_A [6]; the norm (6) is called the "energy" norm and is proportional to the bending energy of the beam, evaluated according to the linearized theory.

By means of some classical results on Sobolev spaces [7] it is easy to prove that the so defined displacements $v(z)$ are continuous with their first derivative and satisfy the following inequalities

$$(7) \qquad \max_{z\in[0,l]} |v(z)| \leq k_1 |||v||| \qquad \max_{z\in[0,l]} |v'(z)| \leq k_2 |||v|||$$

$$||v|| = \left(\int_0^l v^2 \, dz\right)^{1/2} \leq k_3 |||v||| \qquad ||v'|| = \left(\int_0^l v'^2 \, dz\right)^{1/2} \leq k_4 |||v|||.$$

The potential energy $E(v)$ of the bar, given by (3), can be of course defined in the set of the displacement functions $v(z)$ such that

$$(8) \qquad v'^2(z) \leq \beta < 1$$

that is, taking into account the second inequality (7), in the subset of H_A defined by

$$(8') \qquad |||v||| \leq \delta = \frac{\beta^{1/2}}{k_2}.$$

Thus we will consider the functional $E(v)$ defined in the spherical neighborhood S_δ of the origin

$$(9) \qquad S_\delta = \{v \in H_A : |||v||| \leq \delta\}.$$

We can now formulate the following theorem, very useful for the analysis of the Lyapunov stability of the bar:

THEOREM 1. *The energy functional* $E(v)$ *is at least twice Fréchet differentiable at* $v = 0$ *in the space* H_A.

Proof. We will indicate with

$$(10) \qquad D^{(n)} E(v_1; v) = \left[\frac{d^n}{d\alpha^n}\left(\frac{E(v_1 + \alpha v) - E(v_1)}{\alpha}\right)\right]_{\alpha=0}$$

the n^{th} Gâteaux (or weak) differential of $E(v)$ at v_1 in the direction v. Thus the first and the second Gâteaux differentials of $E(v)$ are

$$(11) \qquad D^{(1)} E(v_1; v) = \frac{1}{2} EI \int_0^l \left[\frac{2 v_1'' v''}{1 - v_1'^2} + \frac{2 v_1''^2 v_1' v'}{(1 - v_1'^2)^2} - \frac{2\lambda}{l^2} \frac{v_1' v'}{(1 - v_1'^2)^{1/2}}\right] dz$$

$$(12) \qquad D^{(2)} E(v_1; v) = EI \int_0^l \left[\frac{v''^2}{1 - v_1'^2} + \frac{v_1''^2 v'^2 + 2 v_1' v_1'' (2 v' v'')}{(1 - v_1'^2)^2} + \right.$$

$$\left. + \frac{4 v_1'^2 v_1''^2 v'^2}{(1 - v_1'^2)^3} - \frac{\lambda}{l^2}\left(\frac{v'^2}{(1 - v_1'^2)^{1/2}} + \frac{v_1'^2 v'^2}{(1 - v'^2)^{3/2}}\right)\right] dz.$$

For the second Fréchet differentiability of $E(v)$ at $v = 0$ we have to prove the following condition

$$E(v) = E(0) + D^{(1)} E(0\,;v) + \tfrac{1}{2} D^{(2)} E(0\,;v) + o(|||v|||^2) \tag{13}$$

where:

$$\lim_{|||v|||\to 0} \frac{o(|||v|||^2)}{|||v|||^2} = 0 \tag{14}$$

and $D^{(1)} E(0\,;v)$, $D^{(2)} E(0\,;v)$ are continuous functionals, linear and quadratic in v respectively.

According to (3), (11) and (12) we have

$$\begin{aligned} &E(0) = 0 \\ &D^{(1)} E(0\,;v) = 0 \qquad (15) \\ &|D^{(2)} E(0\,;v)| \leq \left(1 + \frac{\lambda}{l^2} k_4^2\right) |||v|||^2 . \end{aligned}$$

On the other hand the mean value theorem gives

$$\begin{aligned} E(v) &= \tfrac{1}{2} D^{(2)} E(0\,;v) + \tfrac{1}{2} [D^{(2)} E(\alpha v\,;v) - D^{(2)} E(0\,;v)] = \\ &= \tfrac{1}{2} D^{(2)} E(0\,;v) + r_2(\alpha v\,;v) \qquad 0 \leq \alpha \leq 1 . \end{aligned} \tag{16}$$

Hence if we write

$$m(v) = \max_{z\in[0,l]} |v'| \tag{17}$$

we get

$$|r_2(\alpha v\,;v)| = \frac{1}{2} |D^{(2)} E(\alpha v\,;v) - D^{(2)} E(0\,;v)| = \tag{18}$$

$$= \left| \frac{EI}{2} \int_0^l \left\{ \left(\frac{1}{1 - \alpha^2 v'^2} - 1 \right) v''^2 + \frac{5\,\alpha^2 v'^2 v''^2}{(1 - \alpha^2 v'^2)^2} + \frac{4\,\alpha^2 v'^4 v''^2}{(1 - \alpha^2 v'^2)^3} - \right.\right.$$

$$\left.\left. - \frac{\lambda}{l^2} \left[\left(\frac{1}{(1 - \alpha^2 v'^2)^{1/2}} - 1 \right) v'^2 + \frac{\alpha^2 v'^4}{(1 - \alpha^2 v'^2)^{3/2}} \right] \right\} dz \right| \leq$$

$$\leq \frac{|||v|||^2}{2} \left[\left(\frac{1}{1 - m^2} - 1 \right) + \frac{5\,m^2}{1 - m^2} + \frac{4\,m^4}{(1 - m^2)^3} + \right.$$

$$\left. + \lambda \left(\frac{1}{(1 - m^2)^{1/2}} - 1 + \frac{m^2}{(1 - m^2)^{3/2}} \right) \right] .$$

Thus

(19) $$\lim_{|||v|||\to 0} \frac{|r_2(\alpha v\,;v)|}{|||v|||^2} = 0\,.$$

The third inequality (15), which implies the continuity of $D^{(2)}E(0\,;v)$, together with (19), prove the second Fréchet differentiability of $E(v)$ at $v=0$. □

We remark also that the Fréchet differentiability of $E(v)$ at $v = 0$ implies, of course, the continuity of $E(v)$ at $v = 0$.

3. LYAPUNOV STABILITY OF THE RECTILINEAR CONFIGURATIONS OF THE CENTRALLY LOADED BAR

If $\dot v(z,t)\,(\dot{\ }) = d(\ )/dt$, defines the velocity field, the kinetic energy of the bar is given by

(20) $$T(\dot v) = \tfrac{1}{2}\int_0^l \mu \dot v^2\, dz$$

where μ is the mass for unit length of the bar.

A generic state of motion of the bar $\begin{pmatrix} v \\ \dot v \end{pmatrix}$ will be considered as an element of the Hilbert space

(21) $$H_{A_m} = H_A \times L_2$$

whose norm is defined as

(22) $$\left\| \begin{pmatrix} v \\ \dot v \end{pmatrix} \right\| = |||v|||^2 + 2\,T(\dot v)\,.$$

The total potential energy functional E_T is

(23) $$E_T\left(\begin{pmatrix} v \\ \dot v \end{pmatrix}\right) = E(v) + T(\dot v)$$

and is defined in the subset $S_\delta \times L_2$ in the space H_{A_m}. According to the previously proven properties of $E(v)$ we can state that also $E_T\begin{pmatrix} v \\ \dot v \end{pmatrix}$ is a continuous and Fréchet differentiable functional at $\begin{pmatrix} v \\ \dot v \end{pmatrix} = 0\,.$

Let us analyze now the stability of the rectilinear configuration C_0 of the bar. According to the Lyapunov definition, the equilibrium configuration of the bar is stable if

(24) $$\forall \varepsilon > 0\ \exists \delta > 0 : \begin{pmatrix} u(0) \\ \dot u(0) \end{pmatrix} \in H_{A_m}\,, \left\| \begin{pmatrix} u(0) \\ \dot u(0) \end{pmatrix} \right\| < \delta \mapsto \left\| \begin{pmatrix} u(t) \\ \dot u(t) \end{pmatrix} \right\| < \varepsilon \quad \forall t > 0$$

that is, in brief, all the motions that start in the neighborhood of C_0, according to the norm of the space H_{A_m}, remain always near C_0.

After this preliminary definition let us consider the first energy criterion, that is the extension to continuous conservative systems of the Lagrange–Dirichlet theorem.

Namely we want to prove that the following minimum condition

$$\exists \bar{\rho} > 0 : \quad \inf_{\partial S_\rho} E(v) > 0 \qquad \forall \rho : 0 < \rho \leq \bar{\rho} \tag{25}$$

where

$$\partial S_\rho = \{v \in H_A : |||v||| = \rho\} \tag{26}$$

implies Lyapunov stability at the rectilinear configuration C_0 of the bar.

The proof of this statement follows very easily if we take $E_T\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix}$ as a Lyapunov functional and we take into account the continuity of $E(v)$ at $v = 0$.

Condition (25) represents therefore a sufficient condition of Lyapunov stability. Let us now formulate the energy criterion of stability that involves the second differential of $E(v)$ and that corresponds to the most usual formulation of the so called energy criterion.

THEOREM 2. *The rectilinear configurations* C_0 *of the centrally loaded bar are Lyapunov stable if the second differential of* $E(v)$ *at* C_0 *is such that*

$$\omega = \min_{v \in H_A - \{0\}} \frac{D^{(2)} E(0; v)}{|||v|||^2} > 0. \tag{27}$$

The statement of Theorem 2 corresponds to saying that the rectilinear configurations of the bar are Lyapunov stable for $N < N_c$ *if* N_c *represents the Euler (buckling) load.*

Proof. At first taking into account the expression of $D^{(2)} E(0; v)$

$$D^{(2)} E(0; v) = EI \int_0^l \left(v''^2 - \frac{\lambda}{l^2} v'^2\right) dz \tag{28}$$

it is easy to prove the existence, by means of the Sobolev "embedding" theorem, of a displacement $v(z)$ in the space H_A at which the ratio $D^{(2)} E(0; v)/|||v|||^2$ attains its minimum. If this minimum ω is positive the equilibrium is Lyapunov stable. In fact, because $E(v)$ is twice Fréchet differentiable, we have

$$E(v) = \tfrac{1}{2} D^{(2)} E(0; v) + o(|||v|||^2) \tag{29}$$

taking into account the equilibrium of the rectilinear configuration of the bar. Now if we indicate by v^* the displacements with $|||v||| = 1$, we can also write:

$$E(v) = |||v|||^2 \left[\frac{1}{2} D^{(2)} E(0; v^*) + \frac{o(|||v|||^2)}{|||v|||^2}\right] \geq \tag{30}$$

$$\geq |||v|||^2 \left[\omega - \frac{|o(|||v|||^2)|}{|||v|||^2}\right].$$

Now because

$$\forall \varepsilon > 0 \quad \exists \rho > 0 : |||v||| < \rho \Rightarrow \left|\frac{o(|||v|||^2)}{|||v|||^2}\right| < \varepsilon \tag{31}$$

if we take ε such that $\omega - \varepsilon > 0$ we have

$$\exists \rho > 0 : 0 < |||v||| < \rho \Rightarrow E(v) > (\omega - \varepsilon)\, |||v|||^2 > 0 \tag{32}$$

and, because of (25), the statement of the theorem is proven. □

Now it is well known that the critical load N_c is the smallest value of λ that gives $\omega = 0$, i.e. from eq. (28)

$$\frac{1}{\lambda_c} = \max_{v \in H_A - \{0\}} \frac{\int_0^l v'^2 \, dz}{\int_0^l v''^2 \, dz}. \tag{33}$$

Therefore for $0 \leq \lambda < \lambda_c$ it is $1 \geq \omega > 0$ and we have Lyapunov stability. For instance if the column has both hinged ends for $N < N_c = \dfrac{\pi^2 EI}{l^2}$ the equilibrium configuration is Lyapunov stable.

4. CONNECTION BETWEEN THE NORM OF THE INITIAL DISTURBANCE AND THE NORM OF THE PERTURBED MOTION

In order to characterize more accurately the meaning of the stability of the rectilinear configuration of the bar for $\lambda < \lambda_c$, let us pass on to examine the connection between the amplitude of the initial disturbance and the maximum amplitude, in the sense of the norm in H_{A_m}, of the disturbed motion of the bar.

It is at first useful to evaluate the constants of the inequalities (7); for instance if the column has one end built in and the other free, we have

$$K_1 = l \left(\frac{l}{EI}\right)^{1/2} \qquad K_2 = \left(\frac{l}{EI}\right)^{1/2}$$

$$K_3 = l^{3/2} \left(\frac{l}{EI}\right)^{1/2} \qquad K_4 = l^{1/2} \left(\frac{l}{EI}\right)^{1/2}. \tag{7'}$$

Inequalities (7), together with the properties of differentiability of the potential energy $E(v)$ of the bar, enable to establish upper and lower bounds on the total energy $E_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix}$. In fact taking into account the definition (32) of the critical load λ_c and eqs (17), (18), (33), we get the following lower bound on $E(v)$:

$$E(v) = \frac{1}{2} D^{(2)} E(0 ; v) + r_2(\alpha v ; v) \geq \tag{34}$$

$$\geq \frac{1}{2} |||v|||^2 \left(1 - \frac{\lambda}{\lambda_c}\right) + r_2(\alpha v ; v) \geq$$

$$\geq \frac{1}{2} |||v|||^2 \left\{1 - \frac{\lambda}{\lambda_c} - \frac{\lambda}{\lambda_c} 2 \left[\frac{1}{(1 - m^2)^{1/2}} - 1 + \frac{m^2}{(1 - m^2)^{3/2}}\right]\right\}.$$

An upper bound to $E(v)$ can be directly derived from the expression (3) of the potential energy:

$$E(v) \leq \frac{1}{2} |||v|||^2 \frac{1}{1 - m^2}. \tag{35}$$

Thus we have

$$\frac{1}{2} |||v|||^2 \left(1 - \frac{\lambda}{\lambda_c} f(m)\right) \leq E(v) \leq \frac{1}{2} |||v|||^2 \frac{1}{1 - m^2} \tag{36}$$

where

$$f(m) = 1 + 2 \left[\frac{1}{(1 - m^2)^{1/2}} - 1 + \frac{m^2}{(1 - m^2)^{3/2}}\right]. \tag{37}$$

The total energy $E_t \left(\begin{pmatrix} v \\ \dot{v} \end{pmatrix}\right)$ satisfies then the following inequality:

$$\frac{1}{2} |||v|||^2 \left(1 - \frac{\lambda}{\lambda_c} f(m)\right) + T(\dot{v}) \leq E_t \left(\begin{pmatrix} v \\ \dot{v} \end{pmatrix}\right) \leq \frac{1}{2} \left\| \begin{pmatrix} v \\ \dot{v} \end{pmatrix} \right\|^2 \frac{1}{1 - m^2}. \tag{38}$$

Let now the beam be subjected, at $t = 0$, to a disturbance $\begin{pmatrix} v(0) \\ \dot{v}(0) \end{pmatrix}$ with:

$$\left\| \begin{matrix} v(0) \\ \dot{v}(0) \end{matrix} \right\| \leq \delta \tag{39}$$

we have also from (7) (7′) (17) and (39):

$$m^2(0) \leq \frac{l}{EI} \delta^2 \tag{40}$$

and

$$\frac{1}{2} \left\| \begin{pmatrix} v(0) \\ \dot{v}(0) \end{pmatrix} \right\|^2 \frac{1}{1 - m^2(0)} \leq \frac{1}{2} \frac{\delta^2}{1 - \frac{l}{EI} \delta^2}. \tag{41}$$

The inequality (38) gives then the condition

$$m^2(t)\left(1-\frac{\lambda}{\lambda_c}f(m(t))\right)\leq\frac{\frac{l}{EI}\delta^2}{1-\frac{l}{EI}\delta^2} \tag{42}$$

which represents a bound on the motion consequent to the initial disturbance.

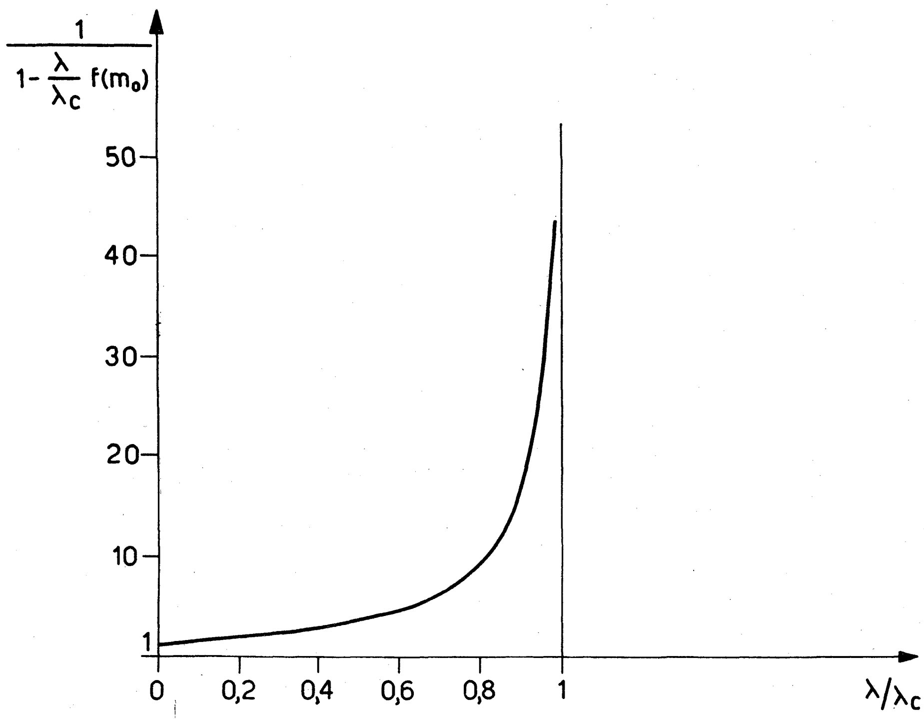

Fig. 1.

In fact if $F(m, \lambda)$ is the function:

$$F(m, \lambda) = m^2\left(1-\frac{\lambda}{\lambda_c}f(m)\right) \tag{43}$$

let, for every value of λ/λ_c, $m_0 = m_0(\lambda)$ be the value of m that gives the maximum value F_{max} of $F(m, \lambda)$. When $m \leq m_0$ the function $F(m, \lambda)$ is positive and therefore if the initial disturbance is such that

$$\frac{\delta^2}{1-\frac{l}{EI}\delta^2}\leq F_{max} \tag{44}$$

we deduce that the consequent motion $\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix}$ satisfies the conditions

$$m(t) \leq m_0 \tag{45}$$

$$1 - \frac{\lambda}{\lambda_c} f(m(t)) \geq 1 - \frac{\lambda}{\lambda_c} f(m_0) > 0 .$$

The inequalities (38) and (45) then give the following simple bounds on the energy norm of $v(t)$:

$$|||v(t)|||^2 \leq \frac{\delta^2}{1 - \frac{l}{\mathrm{EI}}\delta^2} \frac{1}{1 - \frac{\lambda}{\lambda_c} f(m_0)} \tag{46}$$

and on the kinetic energy $\mathrm{T}(\dot{v})$:

$$\mathrm{T}(\dot{v}(t)) \leq \frac{1}{2} \frac{\delta^2}{1 - \frac{l}{\mathrm{EI}}\delta^2} . \tag{47}$$

In eq. (46) when λ approaches the critical value λ_c, the term $1/\left(1 - \frac{\lambda}{\lambda_c} f(m_0)\right)$ diverges and the bound on the amplitude of the disturbed motion fails. The fig. 1 gives the numerical values of the factor $1/\left(1 - \frac{\lambda}{\lambda_c} f(m_0)\right)$, as function of λ/λ_c.

We remark that the inequality (46) provides only an upper bound but not necessarily the least upper bound on the disturbed motions.

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