Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Rolf Reissig

Periodic solutions of $(a(t)x')' + f(t,x) = q(t)$

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **61** (1976), n.1-2, p. 54–61.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1976_8_61_1-2_54_0>

Equazioni differenziali ordinarie. — *Periodic solutions of* $(a(t)x')' + f(t, x) = q(t)$. Nota [(*)] di Rolf Reissig, presentata dal Socio G. Sansone.

RIASSUNTO. — Questa Nota dà alcune estensioni di recenti risultati di S. H. Chang e J. Mawhin–K. Schmitt sulle soluzioni periodiche dell'equazione $(a(t)x')' + f(t, x) = q(t)$ i cui termini sono periodici in t. Sono dimostrati alcuni teoremi di esistenza per mezzo del teorema di Leray–Schauder collegato con applicazioni dispari sugli spazi di Banach.

Let

$$X_k = \{x(t) \in C^k(R) : x(t+T) \equiv x(t)\}, \qquad k \in N$$

and define the Banach space

$$X = (X_0, \|x\| = \max |x(t)|).$$

Consider the differential equation

$$(a(t)x')' + f(t, x) = q(t) \tag{1}$$

where

$$a(t) \in X_1 ; 0 < a_0 = \min a(t), \; a_1 = \max a(t)$$

$$q(t) \in X_0 ; Q(t) = \int_0^t q(t)\, dt \in X_1, \quad \text{i.e. } Q(T) = 0$$

$$f(t + T, x) \equiv f(t, x) \in C^0(R^2).$$

Let $x^*(t) \in X_2$ be the periodic solution of

$$(a(t)x')' = q(t)$$

with a vanishing initial value:

$$x^*(t) = \int_0^t (a(t))^{-1} Q(t)\, dt - \theta^{-1} \int_0^T (a(t))^{-1} Q(t)\, dt \int_0^t (a(t))^{-1}\, dt$$

$$\left[\theta = \int_0^T (a(t))^{-1}\, dt\right].$$

Let us formulate some statements concerning the oscillation problem of (1).

(*) Pervenuta all'Accademia il 19 agosto 1976.

THEOREM 1. *Let there exist a constant* $\mu > 0$ *such that*

$$\int_0^T f(t, -x(t))\,dt \int_0^T f(t, x(t))\,dt < 0 \tag{2}$$

for all $x(t) \in X_2$ *such that* $\min |x(t)| \geqq \mu$. *Let there exist a constant* $k > 0$ *such that*

$$|f(t,x)| \leqq k(T\theta)^{-1} \tag{3}$$

when $|x| \leqq h = 2\|x^*\| + \mu + k$.

Then equation (1) *has at least one* T-*periodic solution.*

THEOREM 2. *Assume condition* (2) *holds as well as*

$$\frac{|f(t,x)|}{|x|} \leqq k_0 < \pi^2 (a_1 \theta^2)^{-1} \tag{4}$$

when $|x| \geqq \mu$.

Then equation (1) *has at least one* T-*periodic solution.*

THEOREM 3. *Let there exist a constant* $\mu > 0$ *such that*

$$x f(t,x) \leqq 0 \tag{5}$$

when $|x| \geqq \mu$.

Then equation (1) *has at least one* T-*periodic solution.*

Note 1. Since $\theta \leqq a_0^{-1} T$ condition (3) is satisfied when

$$|f(t,x)| \leqq k a_0 T^{-2}.$$

Condition (2) is equivalent to both

$$\int_0^T f(t, x(t))\,dt \neq 0 \tag{6'}$$

for all $x(t) \in X_2$ such that $|x(t)| \geqq \mu$, and

$$\int_0^T f(t, -\mu)\,dt \int_0^T f(t, \mu)\,dt < 0. \tag{6''}$$

Indeed, when $x(t) \in X_2$ and $x(t) \geqq \mu$ consider the continuous function

$$I(\lambda) = \int_0^T f(t, (1-\lambda)\mu + \lambda x(t))\,dt, \qquad 0 \leqq \lambda \leqq 1.$$

If $\operatorname{sgn} I(0) = \varepsilon = -\operatorname{sgn} I(1)$ then there is a $\lambda_0 \in (0, 1)$ such that $I(\lambda_0) = 0$, in contradiction to (6′). Consequently

$$\operatorname{sgn} \int_0^T f(t, x(t))\, dt = \varepsilon.$$

A similar argument yields that

$$\operatorname{sgn} \int_0^T f(t, -x(t))\, dt = -\varepsilon.$$

Theorem 1 improves a result of J. Mawhin-K. Schmitt [5] where it is assumed that conditions (6′)-(6″) hold as well as

$$|f(t, x)| \leqq k a_0 (2\, T^2)^{-1} \qquad \text{when} \quad |x| \leqq h. \tag{7}$$

Note 2. Condition (2) is satisfied when

$$x f(t, x) > 0 \qquad (|x| \geqq \mu). \tag{8′}$$

Using a standard limit process we can show that the existence of a T-periodic solution is ensured even under condition (3) and the weaker version of (8′)

$$x f(t, x) \geqq 0 \qquad (|x| \geqq \mu). \tag{8″}$$

Together with Theorem 3. this improves a result of S. H. Chang [1] where (8″) or (5) is proposed as well as

$$|f(t, x)| \leqq \frac{k}{3} \min(1, a_0 T^{-2}) \qquad \text{when} \quad |x| \leqq h. \tag{9}$$

Note 3. Conditions (3), (7) and (9) are satisfied when

$$\lim_{t \to \infty} \frac{|f(t, x)|}{|x|} = 0 \qquad \text{(uniformly in } t). \tag{10}$$

Assume (10) and define for all $r \geqq 0$.

$$F(r) = \max \{ |f(t, x)| : \quad 0 \leqq t \leqq T, |x| \leqq r \}.$$

Let $\varepsilon > 0$ be arbitrarily chosen; then $|f(t, x)| \leqq \varepsilon |x|$ when $|x| \geqq h(\varepsilon)$. Let $r \geqq r_0 = h(\varepsilon)$; then

$$F(r) \leqq F(r_0) + \varepsilon r \quad , \quad \limsup_{r \to \infty} \frac{F(r)}{r} \leqq \varepsilon ,$$

hence

$$\lim_{r \to \infty} \frac{F(r)}{r} = 0.$$

Let c and h_0 be arbitrary positive constants. Since

$$\frac{F(r)}{r} \leqq \frac{c}{2} \qquad \text{if} \quad r \geqq r^* = h_0 + k \quad (k \text{ sufficiently large})$$

we obtain

$$|f(t, x)| \leqq \frac{c}{2} r^* \leqq ck \qquad \text{for} \quad 0 \leqq t \leqq T, |x| \leqq r^*$$

provided that $k \geqq h_0$. Finally identify c with the corresponding value of conditions (3), (7) or (9) and h_0 with $2\|x^*\| + \mu$.

The result of J. Mawhin-K. Schmitt is based on an abstract coincidence theorem whereas S. H. Chang uses Lazer's fixed point technique developed for the oscillation problem of Liénard's equation. However, our proof of Theorem 1-3 is a simple application of Leray-Schauder degree in connection with odd mappings in Banach spaces (see: [2], [3]). For this purpose let us introduce the independent variable

$$\tau = \int_0^t (a(t))^{-1} \, dt \; [\theta = \tau(T)];$$

denote

$$x(t) = x \circ t(\tau) = \xi(\tau) \; [x(t+T) = x \circ t(\tau + 0) = \xi(\tau + \theta)],$$

$$f(t, \xi) = f(t(\tau), \xi) = \varphi(\tau, \xi),$$

$$a(t) = a \circ t(\tau) = \alpha(\tau) = \frac{dt}{d\tau} \left[\int_0^\theta \alpha(\tau) \, d\tau = T \right],$$

$$a(t) q(t) = \alpha(\tau) q \circ t(\tau) = \rho(\tau) = P'(\tau) \left[P(\tau) = \int_0^\tau \rho(\tau) \, d\tau = Q \circ t(\tau) \right].$$

Differential equation (1) is transformed into

(11) $$\xi'' + \alpha(\tau) \varphi(\tau, \xi) = \rho(\tau).$$

The θ-periodic solution with initial value zero of the differential equation

$$\xi'' = \rho(\tau)$$

is

$$\xi^*(\tau) = \int_0^\tau P(\tau) \, d\tau - \frac{\tau}{\theta} \int_0^\theta P(\tau) \, d\tau = x^* \circ t(\tau).$$

Consider the comparison system

(12) $$\xi'' + \Phi(\tau, \xi, \lambda) = \rho(\tau)$$

where $\lambda \in [0, 1]$ is a parameter and

$$\Phi(\tau, \xi, \lambda) = \frac{\alpha(\tau)}{2} [(1 + \lambda) \varphi(\tau, \xi) - (1 - \lambda) \varphi(\tau, -\xi)],$$

$$\Phi(\tau, -\xi, 0) = -\Phi(\tau, \xi, 0), \Phi(\tau, \xi, 1) = \alpha(\tau) \varphi(\tau, \xi).$$

It is a well-known fact (see: [2], [3]) that there are periodic solutions of (12) for all $\lambda \in [0, 1]$ when there exists a ball $B_r = \{x(t) \in X : \|x\| < r\}$ such that $x(t) = \xi \circ \tau(t) \notin \partial B_r$ for every periodic solution of (12). For an arbitrary $\lambda \in [0,1]$ at least one periodic solution $\xi(\tau)$ of (12) is such that $x(t) = \xi \circ \tau(t) \in B_r$. Such a ball exists, for instance, when all periodic solutions of (12) are a priori bounded, and that uniformly with respect to λ. In this special case every ball B_r the radius r of which is sufficiently large is adequate.

Let $\xi(\tau)$ be a θ-periodic solution of (12); then

$$\int_0^\theta \Phi(\tau, \xi(\tau), \lambda) \, d\tau = 0,$$

i.e.

$$(1 + \lambda) \int_0^\theta \alpha(\tau) \varphi(\tau, \xi(\tau)) \, d\tau = (1 - \lambda) \int_0^\theta \alpha(\tau) \varphi(\tau, -\xi(\tau)) \, d\tau$$

or $[\xi \circ \tau(t) = x(t)]$

$$(1 + \lambda) \int_0^T f(t, x(t)) \, dt = (1 - \lambda) \int_0^T f(t, -x(t)) \, dt.$$

If $|x(t)| \geq \mu$ for all t then we obtain a contradiction to (2). Consequently there is a $\tau_0 \in [0, \theta]$ such that $|\xi(\tau_0)] < \mu$. Let

$$\xi(\tau) = \sigma(\tau) + \zeta(\tau), \sigma(\tau) = [\xi^*(\tau) - \xi^*(\tau_0)] + \xi(\tau_0)$$

$$[\|\sigma\| < 2\|x^*\| + \mu];$$

then

$$\zeta'' + \Phi(\tau, \sigma(\tau) + \zeta, \lambda) = 0. \tag{13}$$

Assume that $\|\xi\| = h$; multiplying equation (13) by $\zeta(\tau)$ and integrating from zero to θ we obtain

$$\int_0^\theta (\zeta'(\tau))^2 \, d\tau = \int_0^\theta \zeta(\tau) \Phi(\tau, \sigma(\tau) + \zeta(\tau), \lambda) \, d\tau \tag{14}$$

$$\leq k(T\theta)^{-1} \int_0^\theta \alpha(\tau) |\zeta(\tau)| \, d\tau$$

$$\leq k\theta^{-1} \|\zeta\|.$$

If $\tau_0 \leqq \tau \leqq \tau_0 + \theta$ we further conclude that

$$\zeta^2(\tau) = \left(\int_{\tau_0}^{\tau} \zeta'(\tau)\,d\tau\right)^2 \leqq \theta \int_0^{\theta} (\zeta'(\tau))^2\,d\tau \leqq k\,\|\zeta\|,$$

i.e.

$$\|\zeta\| \leqq k \quad \text{and} \quad \|\xi\| \leqq \|\sigma\| + \|\zeta\| < h$$

in contradiction to the assumption. There is no solution $\xi(\tau)$ of (12) such that $x(t) = \xi \circ \tau(t) \in \partial B_r \subset X$, $r = h$.

COROLLARY 1. Condition (2) can be replaced by condition (8″); this is obvious when the strict version (8′) is valid (see: Note 2). Consider the differential equation

$$(a(t)x')' + [\vartheta\rho x + (1-\vartheta)f(t,x)] = q(t) \tag{15}$$

where

$$0 < \vartheta \leqq 1, \qquad \rho = h^{-1}k(T\theta)^{-1}.$$

Note that

$$|\vartheta\rho x + (1-\vartheta)f(t,x)| \leqq \vartheta k(T\theta)^{-1} + (1-\vartheta)\,|f(t,x)| \leqq k(T\theta)^{-1}$$

when $|x| \leqq h$; furthermore note that

$$x[\vartheta\rho x + (1-\vartheta)f(t,x)] = \vartheta\rho x^2 + (1-\vartheta)\,xf(t,x) \geqq \vartheta\rho x^2 > 0$$

when $|x| \geqq \mu$. Hence, the conditions of Theorem 1 are satisfied, the existence of a periodic solution with a norm smaller than h is ensured. Choose a monotone-decreasing positive sequence $(\vartheta_n)_{n\in N}$ such that $\lim_{n\to\infty} \vartheta_n = 0$; consider a corresponding sequence of periodic solutions of (15) where $\vartheta = \vartheta_n : (x_n(t))_{n\in N}$, $\|x_n\| < h$. There are uniform bounds of the first and second derivatives:

$$\|x_n'\| \leqq h', \quad \|x_n''\| \leqq h''.$$

Taking account of the theorem of Arzelà–Ascoli we may assume that

$$\lim_{n\to\infty} x_n(t) = x(t), \quad \lim_{n\to\infty} x_n'(t) = x'(t)$$

(uniformly on $[0, T]$). Since

$$a(t)x_n'(t) - a(0)x_n'(0) = Q(t) - \vartheta_n \rho \int_0^t x_n(t)\,dt$$
$$- (1-\vartheta_n)\int_0^t f(t, x_n(t))\,dt$$

we obtain the equation

$$a(t)\,x'(t) - a(\mathrm{o})\,x'(\mathrm{o}) = \mathrm{Q}(t) - \int_0^t f(t\,,x(t))\,\mathrm{d}t$$

yielding differential equation (1).

Proof of Theorem 2. Let us mention that, by virtue of (4),

$$|f(t\,,x)| \leqq k_0\,|x| + k_1 \qquad \text{for all} \quad x \in \mathrm{R}$$

$$[k_1 = \max\{|f(t\,,x) - k_0\,x| : \quad \mathrm{o} \leqq t \leqq \mathrm{T}\,,\,|x| \leqq \mu\}].$$

This time we derive from (14):

$$\int_0^\theta (\zeta'(\tau))^2\,\mathrm{d}\tau \leqq \int_0^\theta |\zeta(\tau))|\,\alpha(\tau)\,(k_0\,|\xi(\tau)| + k_1)\,\mathrm{d}\tau$$

$$\leqq k_0\,a_1 \int_0^\theta \zeta^2(\tau)\,\mathrm{d}\tau + \int_0^\theta \alpha(\tau)\,|\zeta(\tau)|\,(k_0\,|\sigma(\tau)| + k_1)\,\mathrm{d}\tau$$

$$\leqq k_0\,a_1 \int_0^\theta \zeta^2(\tau)\,\mathrm{d}\tau + a_1\,[k_0\,(2\,\|x^*\| + \mu) + k_1] \int_0^\theta |\zeta(\tau)|\,\mathrm{d}\tau.$$

A consequence of $\zeta(\tau_0) = \zeta(\tau_0 + \theta) = \mathrm{o}$ is the inequality (see: [6]):

$$\int_0^\theta \zeta^2(\tau)\,\mathrm{d}\tau \leqq \frac{\theta^2}{\pi^2} \int_0^\theta (\zeta'(\tau))^2\,\mathrm{d}\tau \tag{16}$$

by means of which

$$\left(\mathrm{I} - \frac{k_0\,a_1\,\theta^2}{\pi^2}\right)\left(\int_0^\theta \zeta^2(\tau)\,\mathrm{d}\tau\right)^{1/2} \leqq a_1\,[k_0\,(2\,\|x^*\| + \mu) + k_1]\,\sqrt{\theta}\,.$$

A priori bounds for $\int_0^\theta (\zeta'(\tau))^2\,\mathrm{d}\tau$, $\|\zeta\|$ and $\|\xi\|$ are available now. Note once more that condition (2) can be replaced by condition (8″).

Proof of Theorem 3. Note that, by virtue of condition (5),

$$xf(t\,,x) \leqq k_2 \qquad \text{for all} \quad x \in \mathrm{R}$$

$[k_2 = \max\{xf(t\,,x) : \quad \mathrm{o} \leqq t \leqq \mathrm{T}\,,\,|x| \leqq \mu\}]$.

To begin with assume the strict estimate

$$xf(t\,,x) < \mathrm{o}\,(|x| \geqq \mu)$$

from which condition (2) arises. Using a periodic solution $\xi(\tau)$ of (12) we calculate:

$$\int_0^\theta (\xi'(\tau))^2 \, d\tau = \int_0^\theta \xi(\tau)\,[\Phi(\tau, \xi(\tau), \lambda) - \rho(\tau)]\, d\tau$$

$$\leqq k_2 T - \int_0^\theta \xi(\tau)\,\rho(\tau)\, d\tau .$$

Let $\zeta(\tau) = \xi(\tau) - \xi(\tau_0)$ where $|\xi(\tau_0)| \leqq \mu$ $(\|\xi\| \leqq \|\zeta\| + \mu)$; then

$$\int_0^\theta (\zeta'(\tau))^2 \, d\tau \leqq (k_2 + \mu \|q\|)\, T + a_1 \|q\| \sqrt{\theta} \left(\int_0^\theta \zeta^2(\tau)\, d\tau\right)^{1/2}.$$

Applying inequality (16) we obtain a priori bounds for

$$\int_0^\theta \zeta^2(\tau)\, d\tau \quad , \quad \int_0^\theta (\zeta'(\tau))^2\, d\tau \qquad \text{and} \quad \|\zeta\| .$$

The weaker condition (5) being proposed a similar extension of the result as in Corollary 1 is possible.

REFERENCES

[1] S. H. CHANG (1976) – *Periodic solutions of* $(a(t)x')' + f(t, x) = p(t)$, « Quart. J. Math. Oxford », *26*, 105–109.

[2] G. GÜSSEFELDT (1968) – *Der topologische Abbildungsgrad für vollstetige Vertorfelder zum Nachweis von periodischen Lösungen*. « Math. Nachr. », *36*, 231–233.

[3] G. GÜSSEFELDT (1971) – *Zwei Methoden zum Nachweis von periodischen Lösungen bei gewöhnlichen Differentialgleichungen*, « Math. Nachr. », *48*, 141–151.

[4] A. C. LAZER (1968) – *On Schauder's fixed point theorem and forced second-order nonlinear oscillations*, « J. Math. Anal. Appl. », *21*, 421–425.

[5] J. MAWHIN and K. SCHMITT (1976) – *Rothe and Altman type coincidence theorems and applications to differential equations*, « Rapport » *91*, « Sém. Math. Appl. Méc. Univ. Cath », Louvain.

[6] R. REISSIG (1975) – *Extensions of some results concerning the generalized Liénard equation*, « Ann. Mat. Pura Appl. », *104* (4), 269–281.