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On the junctional equation $p(f(z)) = a(z) \sin \alpha(z) + b(z)$

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Analisi funzionale. — *On the functional equation* $p(f(z)) = a(z) \sin \alpha(z) + b(z)$. Nota (*) di CHUNG-CHUN YANG, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia $p(z)$ un polinomio non lineare, $\alpha(z)$ un polinomio non costante, oppure una trascendente intera di ordine finito, e $a(z)$, $b(z)$ due polinomi non costanti di grado inferiore a quello di α (se $\alpha(z)$ è un polinomio). In questa Nota si dànno allora condizioni necessarie perchè l'equazione funzionale $p(f(z)) = a(z) \sin \alpha(z) + b(z)$ abbia per soluzione una trascendente intera $f(z)$.

INTRODUCTION

Let $p(z)$ be a nonlinear polynomial, $a(z)$ and $b(z)$ be two nonconstant polynomials, and $\alpha(z)$ be a trascendental entire function of finite order or a polynomial of degree higher than $\max\{\deg a(z), \deg b(z)\}$. In this note we shall derive a necessary condition when the following equation

$$p(f(z)) = a(z) \sin \alpha(z) + b(z) \tag{1}$$

has a transcendental entire solution $f(z)$. We assume the reader is familiar with the fundamental concepts of Nevanlinna's theory of meromorphic functions and the symbols such as $\bar{N}(r, 0, f)$, $T(r, 0, f)$ and $\delta(a, f)$, etc. (See [2]).

THEOREM. *Let p, a, b, be defined as in the above. In order for equation (1) to have a transcendental entire solution $f(z)$, it is necessary that $p(z) \equiv p_0 (z - a_1)^{n_1} (z - a_2)^{n_2} q^2(z) - c$ and $a(z) \equiv \pm b(z) + c$, where p_0, a_1, a_2 are constants, n_1, n_2 are either 0 or 1, $q(z)$ is a polynomial, and c is a constant.*

Proof. Suppose that $f(z)$ is transcendental entire and satisfies equation (1). Then from (1) we have

$$T(r, p(f)) \sim T(r, \sin \alpha(z)) \quad \text{as } r \to \infty. \tag{2}$$

Assume, for any constant d,

$$a(z) \not\equiv \pm b(z) + d. \tag{3}$$

We now show that for any constant c

$$\overline{\lim_{r\to\infty}} \bar{N}(r, c, a\sin \alpha + b) / N(r, c, a\sin \alpha + b) = 1 \tag{4}$$

(*) Pervenuta all'Accademia il 20 luglio 1976.

possibly outside a set of r values of finite length. Here

$$\bar{\mathrm{N}}(r, a, f) = \int_0^r [\bar{n}(t, a, f) - \bar{n}(0, a, f)]/t \, \mathrm{d}t + \bar{n}(0, a, f) \log r,$$

and $\bar{n}(r, a, f)$ denotes the number of distinct roots of $f(z) = a$ in $|z| \leq t$. Then a zero of

$$a(z) \sin \alpha(z) + b(z) = c \tag{5}$$

with multiplicity $k \geq 2$ is also a zero of

$$a'(z) \sin \alpha(z) + a(z) \alpha'(z) \cos \alpha(z) + b'(z) = 0 \tag{6}$$

with multiplicity $k - 1$. Solving equations (5) and (6) for $\sin \alpha(z)$ and $\cos \alpha(z)$ we obtain

$$\sin \alpha(z) = [c - b(z)]/a(z), \tag{7}$$

and

$$\cos \alpha(z) = [-a(z) b'(z) + a'(z) b(z) - a'(z) c]/a^2(z) \alpha'(z). \tag{8}$$

From the above two equations, it is easy to see that the number of zeros of equation (5) with multiplicities ≥ 2 in a disc $|z| \leq t$ is no greater than twice the number of the roots of the equation

$$\mathrm{H}(z) \equiv \left[\frac{c - b(z)}{a(z)}\right]^2 + \left[\frac{-a(z) b'(z) + a'(z) b(z) - a'(z) c}{a^2(z) \alpha'(z)}\right]^2 = 1 \tag{9}$$

or the equation

$$\begin{aligned} \mathrm{F}(z) \equiv \alpha'^2(z) \{a^2(z) - (c + b(z))^2\} + \\ + \{a'(z) b(z) - a(z) b'(z) - a'(z) c\}^2 = 0 \end{aligned} \tag{10}$$

in the disc $|z| \leq t$, unless $\mathrm{H}(z) \equiv 1$ or $\mathrm{F}(z) \equiv 0$. Clearly $\mathrm{F}(z) \not\equiv 0$ if condition (3) holds. Also in this case we have

$$\mathrm{N}\left(r, \frac{1}{\mathrm{F}}\right) \leq 2 \mathrm{T}(r, \alpha') + \mathrm{O}(1) \log r. \tag{11}$$

Since

$$\mathrm{T}(r, \alpha') = o\, \mathrm{T}(r, \sin \alpha) \quad \text{as} \quad r \to \infty \tag{12}$$

possibly outside a set of r values of finite length, see e.g. [2, p. 54 and p. 55], it follows that

$$\mathrm{N}(r, 1, \mathrm{H}(z)) = o\{\mathrm{T}(r, \sin \alpha)\} \tag{13}$$

as $r \to \infty$ possibly outside a set of r values of finite length. Thus (4) is proved.

By properly choosing the constant c_0 and replacing f by $g+c_0$ in equation (1), we have

(14) $$g^2(z)\{q_n g^{n-2}(z) + q_{n-1} g^{n-3} + \cdots + q_2\} + q_0 = a(z)\sin\alpha(z) + b(z),$$

where n is the degree of p and $q_n, q_{n-1}, \cdots, q_2, q_0$ are constants (among the q's, at least two of them are nonzero). Let

(15) $$q(g(z)) \equiv g^2(z)\{q_n g^{n-2}(z) + \cdots + q_2\} = a(z)\sin\alpha(z) + b(z) - q_0.$$

Then by equation (4) and the fact that $T(r, q(g(z)) \sim nT(r, g)$, we conclude that

(16) $$N(r, 0, g) = o\,T(r, g) \qquad \text{as} \quad r \to \infty.$$

In fact, if it were not so, then from the obvious relation

$$\overline{N}(r, 0, a\sin\alpha + b - q_0) \geq \overline{N}(r, 0, g)$$

and equation (4) one would get a contradiction. Therefore, equation (16) must hold. Hence, for $i = 1, 2, \cdots, n$,

(17) $$\delta(0, g^i) = 1, \qquad i = 1, 2, \cdots, n.$$

Consequently, for any integer m,

(18) $$\theta_m(0, g^i) = 1, \qquad i = 1, 2, \cdots, n.$$

Here

$$\theta_p = (0, h) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r, 0, h)}{T(r, h)},$$

and

$$N_p(r, 0, h) = \sum_{a_k \neq 0} \log^+ \frac{r}{|a_k|} + \min(\rho_0, p)\log r,$$

where the summation is taken over the different zeros a_k ($\neq 0$) of h counted $\min(\rho_k, p)$ times at a_k, ρ_k ($k = 0, 1, 2, \cdots, p$) being the order of multiplicity of zeros of h at a_k. Also we note that g satisfies.

(19) $$\delta(\infty, g^i) = 1, \qquad i = 1, 2, \cdots, n.$$

Now, dividing equation (14) by $b(z) - q_0$ (which is not identically zero) and rewriting the equation, we obtain

(20) $$\sum_{i=0}^{n} \frac{q_i g^i}{b - q_0} - \frac{a}{2i(b-q)} e^{i\alpha} + \frac{ae^{-i\alpha}}{2i(b-q_0)} = 1.$$

This is a functional equation with coefficients growing slower than g and $e^{i\alpha}$'s. We may also assume without loss of generality that none of the q^i's, $i = 1, 2, \cdots$, is zero. Then we can apply a result of Toda's [3] on the functional equation and derive

$$\sum_{i=2}^{n} \theta_n\left(0, \frac{q_i}{b - q_0} g^i\right) + \theta_n\left(0, \frac{ae^{i\alpha}}{2i(b - q_0)}\right) + + \theta_n\left(0, \frac{-ae^{-i\alpha}}{2i(b - q_0)}\right) \leq n. \tag{21}$$

This leads to

$$n + 1 \leq n, \tag{22}$$

giving a contradiction. Thus we must conclude that

$$a^2(z) - (b(z) + c)^2 \equiv 0 \tag{23}$$

that is

$$a(z) \equiv \pm b(z) \pm c.$$

For simplicity we shall only treat the case $a(z) \equiv b(z) + c$. The equation (1) becomes

$$p(g(z)) = a(z)(\sin \alpha(z) + 1) - c \tag{24}$$

or

$$p^*(g(z)) = a(z)\left[\sin \frac{\alpha(z)}{2} + \cos \frac{\alpha(z)}{2}\right]^2 \tag{25}$$

where $p^* \equiv p + c$. We claim that p^* must have the form stated in the theorem. To see this, let us assume that $p^*(z) = p_0(z - a_1)^{n_1}(z - a_2)^{n_2} \cdots (z - a_k)^{n_k}$ where p_0 is a constant, a_i, $i = 1, 2, \cdots, k$ are distinct complex numbers and n_k are positive integers. Thus $p^*(g) = p_0(g - a_1)^{n_1} \cdots (g - a_k)^{n_k}$. Now

$$[p^*(g)/a(z)]^{\frac{1}{2}} = \sin \frac{\alpha(z)}{2} + \cos \frac{\alpha(z)}{2}$$

implies that $p_0^{1/2}(g - a_1)^{n_1/2} \cdots (g - a_k)^{n_k/2}$ is an entire function. It is easy to see that the multiple roots of $\sin \frac{\alpha(z)}{2} + \cos \frac{\alpha(z)}{2}$ are the zeros of $\alpha'(z)$. So unless some of a_i's is actually a Borel exceptional value of g, no n_i can be greater than 2. However, it is easy to rule out this possibility by applying Borel's uniqueness theorem [1] to equation (24). (Here the assumption of the finiteness of the order of α is used).

Therefore we conclude that $n_i \leq 2$ for $i = 1, 2, \cdots, k$. The assertion that there are at most two a_i's that are simple zeros of $p^*(z)$ follows from the well-known fact that a transcendental entire function F can have no more than 2 finite totally ramified values, the term applied to a value a whenever the equation $F - a = 0$ has at most finitely many simple roots. The theorem is thus proved.

Remark. If $b(z) \equiv a(z) + c$ and $a(z) \equiv t^2(z)$ for some polynomial $t(z)$, then $a \sin\alpha + b \equiv a(1 + \sin\alpha) + c \equiv \left[t\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)\right]^2 = p(f)$, where $p(z) \equiv z^2 + c$ and $f(z) = t(z)\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)$.

We state the following:

Conjecture. Let $a(z)$, $b(z)$ be two non-constant polynomials and α be a transcendental entire function or a polynomial of degree higher than that of a and b's. Let p be a nonlinear polynomial, then in order for the functional equation

$$p(f(z)) = a(z)\sin\alpha(z) + b(z)$$

to have a trascendental entire solution f iff $p(z)$ is quadratic, $a(z) \equiv \pm b(z) + c$ for some constant c, and $a(z) \equiv q^2(z)$ for some polynomial $q(z)$.

REFERENCES

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[2] W. K. HAYMAN (1964) – *Meromorphic functions*, Oxford Math. Monographs, Clarendon Press, Oxford.
[3] N. TODA (1971) – *On the functional equation* $\sum_{i=0}^{p} a_i f_i^{n_i} = 1$, « Tohoku Math. Jour. », *23*.