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Analisi funzionale. — *On some classes of operators.* IX. *Well–bounded operators of order p.* Nota di VASILE I. ISTRĂŢESCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Lo spazio A^p ($1 < p < \infty$) di funzioni definite su un intervallo $[a, b]$ tale che per ogni divisione $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ sia

$$\sum_0^{n-1} \frac{|f(x_{i+1}) - f(x_i)|^p}{|x_{i+1} - x_i|^{p-1}} = R_\Delta(f) < \infty ,$$

con la norma $|||f|||_p^p = \sup_\Delta R_\Delta(f) + \sup |f(x)|^p$, è uno spazio di Banach. In questo lavoro si studiano gli operatori T in A^p aventi la seguente proprietà: esiste un intervallo $[a, b]$ tale che per ogni polinomio $p(\lambda)$ valga l'ineguaglianza $\| p(\tau) \| \leq ||| p(\lambda) |||_p$, e si dà una decomposizione spettrale per questi operatori.

INTRODUCTION

D. R. Smart has considered an interesting class of operators on reflexive Banach spaces which have some properties similar to those of hermitian operators on Hilbert spaces. This class was extended to some operators with complex spectra by J. Ringrose. The class of operators studied by Smart and Ringrose is defined as follows: we say that a bounded operator on a Banach space is well-bounded if it is possible to choose a constant K and a finite interval $J = [a, b]$ of the real line in such a way that

$$\| p(\tau) \| \leq K \left(\sup_{x \in J} | p(x) | + \operatorname{var}_J p(x) \right)$$

for every real polynomial $p(t)$. We may write the above inequality in the form

$$\| p(\tau) \| \leq K \| p \|_J$$

where $\| p \|_J = \sup_{t \in J} | p(\tau) | + \operatorname{var}_J p(t)$. Using an approximation theorem it is easy to see that we can extend the application

$$p \to p(\tau)$$

to the functions which are absolutely continuous on the interval J.

(*) Nella seduta dell'8 maggio 1976.

As is well-known for any absolutely continuous function f on the interval J there exists a function $f_1 \in L^1 [a, b]$ such that

$$(*) \qquad f(z) = \int_a^x f_1(x)\, dt.$$

This remark suggests the consideration of a class of functions wich are representable under the form (*) with $f \in L^p$. Such a class was considered by Fr. Riesz and was completely characterized by him as follows: a function has the form $f(x) = \int_a^x f_1(t)\, dt$ where $f_1 \in L^p [a, b]$ iff, for any dissection

$$a = x_0 < x_1 < \cdots < x_n = b,$$

we have

$$\sum_0^{n-1} \frac{|f(x_{i+1}) - f(x_i)|^p}{|x_{i+1} - x_i|^{p-2}} \leq K < \infty,$$

the constant k depending upon the function f. For a proof see J. P. Natansohn, Teoriia functssi [4], Chap. 9, § 4.

We can introduce a space of functions over $[a, b]$ by defining a norm: $f \in A^p$, a Riesz space, if for any dissection

$$\Delta : a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

$$\sum_0^{n-1} \frac{|f(x_{i+1}) - f(x_i)|^p}{|x_{i+1} - x_i|^{p-1}} \leq K$$

and

$$\sup_\Delta \sum_0^{n-1} \frac{|f(x_{i+1}) - f(x_i)|^p}{|x_{i+1} - x_i|^{p-1}} = Rf.$$

The norm on the Riesz space A^p is

$$f \to |||f|||_p^p = \sup |f(x)|^p + Rf.$$

We have the following structure theorem.

THEOREM 0.1. *A^p, with the norm defined above, is a generalized Banach algebra* (1).

Proof. It is obvious that A^p is a linear space and

$$f \to |||f|||_p$$

(1) We call an algebra A generalized normed algebra if

1) A is normed linear space,

2) for all $x, y \in A$, $\|xy\| \leq k \|x\| \|y\|$, k being independent of x, y.

is a norm. We prove now that A^p is complete and next that it is a generalized Banach algebra (clearly A^p has unit element; of course the operations are defined in an usual manner). The completeness follows by an argument similar to Helly's theorem for functions with uniformly bounded variation.

To prove that A^p is a generalized Banach algebra, we estimate the norm of the product of two elements in A^p.

Let $f, g \in A^p$, then

$$|||fg|||_p = \left(\sup |f(x)g(x)|^p + \sup \sum_0^{n-1} \frac{|f(x_{i+1})g(x_{i+1}) - f(x_i)g(x_i)|^p}{|x_{i+1} - x_i|^{p-1}}\right)^{1/p} \leq$$

$$\leq \left[\frac{1}{2^p} \sup |f(x)g(x)|^p + \sum |f(x_{i+1}) - f(x_i)|^p \, |g(x_{i+1})|^p\right]^{1/p} +$$

$$+ \left[\frac{1}{2^p} \sup |f(x)g(x)|^p + \sum |g(x_{i+1}) - g(x_i)|^p \, |f(x_i)|^p\right]^{1/p} \leq$$

$$\leq 2\left(\frac{1}{2^p} |||f|||_p^p \, |||g|||_p^p + |||f|||_p^p \, |||g|||_p^p\right)^{1/p} \leq$$

$$\leq 2\left(1 + \frac{1}{2^p}\right)^{1/p} |||f|||_p \, |||g|||_p$$

and this proves our assertion.

Remark 0.2. The above arguments suggest the consideration of function spaces which we call Riesz spaces: Let $(\Omega, \mathscr{B})$ be a measurable space and μ be a positive measure on $\mathscr{B}$. A function $f : \mathscr{B} \to R$ is called in $A^p(\Omega, \mathscr{B}, \mu)$ if, for any dissection

$$\Delta = (A_i), \bigcup_1^n A_i = \Omega, \qquad \sum_1^n |f(A_i)|^p \mu(A_i)^{1-p} \leq k.$$

The problem is to give a structure of these spaces, i.e., to obtain an integral representation. For results about this problem see [6].

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1. WELL-BOUNDED OPERATORS OF ORDER p

Using the result in section zero we are in position to consider a class of operators on Banach spaces which represent a generalization of well-bounded operators of Smart.

DEFINITION 1.1. A bounded operator on a Banach space is called well-bounded of order p if there exists a positive constant k and a bounded interval $J = [a, b]$ of the real line such that

$$\| p(\tau) \| \leq k \, ||| p |||_p$$

for every polynomial p.

Remark 1.2. If T is well-bounded of order p, then T^* is also well-bounded of order p.

As an useful result about elements in A^p we need the following approximation theorem.

THEOREM 1.3. *If $f \in A^p$ then there exists a sequence of polynomials $\{p_n\}_{1 \leq n < \infty}$ such that*

$$\lim_{n \to \infty} ||| f - p_n |||_p = 0.$$

Proof. By Riesz's structure theorem quoted above we have that f is of the form

$$f(x) = c + \int_a^x f_1(t) \, dt$$

where $f_1 \in L^p$. By Weierstrass' approximation for L^p spaces we have a sequence of polynomials $\{p_n\}$ such that

$$p_n \xrightarrow{L^p} f .$$

If we define

$$\tilde{p}_n(t) = c + \int_a^x p_n(t) \, dt$$

we prove that $\tilde{p}_n \to f$ in $|||\,,\,|||_p$. Indeed we have, for each dissection

$$a = t_0 < t_1 < \cdots < t_n = b$$

$$| f(t_{i+1}) - f(t_i) - (\tilde{p}_n(t_{i+1}) - \tilde{p}_n(t_i) | = \left| \int_{t_i}^{t_{i+1}} [f_1(t) - p_n(t)] \, dt \right| \leq$$

$$\leq (t_{i+1} - t_i)^{1/q} \left(\int_{t_i}^{t_{i+1}} | f_1 - p_n | \, dt \, ,\right.$$

which gives that

$$\sum_0^{n-1} \frac{| f(t_{i+1}) - f(t_i) - (\tilde{p}_n(t_{i+1}) - \tilde{p}_n(t_i)) |^p}{| t_{i+1} - t_i |^{p-1}} \leq \sum_0^n \int_{t_i}^{t_{i+1}} | f_1 - p_n(t) |^p \, dt \leq$$

$$\leq \int_a^b | f_1 - p_n |^p \, dt \, ,$$

whence the assertion.

Using the above approximation theorem we can prove the following

THEOREM 1.4. *If* τ *is well-bounded of order* p *and* $J = [a, b]$, k *as above, then there exists a unique homomorphism*

$$f \to f(\tau)$$

from the generalized Banach algebra A^p *into* $\mathscr{L}(\mathscr{X})$ *such that:*

1) $f(\tau)$ *has its elementary meaning when* f *is a polynomial,*

2) $\| f(\tau) \| \leq k \||| f |||_p \quad \forall f \in A^p$,

3) $(f(\tau))^* = f(\tau^*) \qquad \forall f \in A^p$ *where* $f \to f(\tau^*)$ *is the correspondence homomorphism from* A^p *into* $\mathscr{L}(\mathscr{X}^*)$.

Proof. For any S we define $f(\tau)$ as the operator obtained in the following manner: since there exist polynomials $\{p_n\}$ such that

$$||| p_n - f |||_p \to 0,$$

then $\{p_n\}$ is a Cauchy sequence in the norm $|||\,,\,|||_p$; since

$$\| p_n(\tau) - p_m(\tau) \| \to 0 \qquad n, m \to \infty,$$

the operator $\lim p_n(\tau)$ is independent of the sequence and we set

$$f(\tau) = \lim p_n(\tau).$$

It is easy to see that the correspondence defined in this way satisfies all our requirements.

For the following result we need a generalization of the notion of decomposition of the identity of order p.

Let $\mathscr{X}$ be a Banach space and $\mathscr{X}^*$ its dual; we write $\langle \varphi, x \rangle$ for the value of the functional $\varphi \in x^*$ at the point x.

DEFINITION 1.5. By decomposition of the identity of order p, for x, we shall mean a family of projections $\{E_\lambda\}_{-\infty<\lambda<\infty}$ such that:

1) there exists real numbers a, b such that

$$E_\lambda = \begin{cases} 0 & \lambda < a \\ I & \lambda \geq b, \end{cases}$$

2) $$E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda \qquad \lambda \leq \mu,$$

3) if $x \in \mathscr{X}$ and $\varphi \in x^*$ then $\langle E_\lambda \varphi, x \rangle$ is a Lebesgue measurable function of λ,

4) there exists a positive constant such that

$$\int_a^b |\langle E_\lambda \varphi , x\rangle|^q \, d\lambda \leq h^q \|\varphi\|^q \|x\|^q ,$$

5) if $x \in \mathscr{X}$ and $\varphi \in \mathscr{X}^*$ and the indefinite integral of $\langle E_\lambda \varphi , x\rangle$ is differentiable on the right at $\lambda = \mu$, then the value of its right-hand derivative at that point is $\langle E_\mu \varphi , x\rangle$,

6) for each fixed $x \in \mathscr{X}$ the linear mapping which sends $\varphi \in \mathscr{X}^*$ into the function $\langle E_\lambda \varphi , x\rangle$ is continuous when $\mathscr{X}^*$ has the weak*-topology and $L^\infty[a , b]$ has its lower weak topology (i.e. the weak*-topology of $[L(a , b)]^* = L^\infty[a , b]$.

Remarks. Our definition differs only slightly from the one used by Ringrose; our alteration is required by the definition of well-bounded operators of order p.

The following theorem represents an extension of Theorem 1 [3] for the case considered by us.

THEOREM 1.6. *In a Banach space let* $\{E_\lambda\}$ *be a decomposition of the identity of order* p. *Then there exists one and only one operator* T *in* $L(\mathscr{X})$ *for which*

$$\langle \varphi , Tx\rangle = b \langle \varphi , x\rangle - \int_a^b \langle E_\lambda \varphi , x\rangle \, d\lambda .$$

Proof. We first prove the existence. Let

$$L(\varphi , x) = b \langle \varphi , x\rangle - \int_a^b \langle E_\lambda \varphi , x\rangle \, d\lambda$$

be a bilinear form on X , X and so we have

$$|L(\varphi , x)| \leq |b| \|\varphi\| \|x\| + \int_a^b |\langle E_\lambda \varphi , x\rangle| \, d\lambda \leq$$

$$\leq \{|b| \|\varphi\| \|x\| + k \|\varphi\| \|x\| (b - a)\}$$

for a fixed element $x \in X$. Thus $L(\varphi , x)$ is a linear functional on X^* and $\int_a^b \langle E_\lambda \varphi , x\rangle \, d\lambda$ is continuous in the corresponding topologies; hence there exists $\gamma = \gamma(x)$ such that

$$L(\varphi , x) = \langle \varphi , \gamma(x)\rangle$$

and

$$x \rightarrow \gamma(x)$$

is linear and continuous since

$$\|y(x)\| \leq \|x\| \tilde{k} .$$

The following result gives a connection between well-bounded operators of order p and the decomposition of identity of order p.

THEOREM 1.7. *Let* $\{E_\lambda\}$ *be a decomposition of identity of order* p *and let* T *be the associated operator as in Theorem 1.6. and let* A^p *be the corresponding algebra for* $J = [a, b]$. *Then,*

1) T *is well-bounded of order* p,

2) *if* $f \to f(\tau)$ *is the homomorphism of Theorem 1.6. then*

$$\langle \varphi, f(\tau) x \rangle = f(b) \langle \varphi, x \rangle - \int_a^b \langle E_\lambda \varphi, x \rangle f'(\lambda)\, d\lambda$$

for all $f \in A^p$ *and* $\varphi \in X^*, x \in \mathscr{X}$.

Proof. We follow the proof by J. R. Ringrose ($p = 1$). We prove by induction that

$$\langle \varphi, T^n x \rangle = b^n \langle \varphi, x \rangle - \int_a^b \langle E_\lambda \varphi, x \rangle n\lambda^{n-1}\, d\lambda$$

for all positive integers n. When $n = 1$ this is clear. We assume that it is true for n and we prove it for $n + 1$. We have then

$$\langle \varphi, T^{n+1} x \rangle = \langle \varphi, T(T^n x) \rangle = b \langle \varphi, T^n x \rangle - \int_a^b \langle E_\lambda \varphi, T^n x \rangle\, d\lambda =$$

$$= b \left\{ b^n \langle \varphi, x \rangle - \int_a^b \langle E_\lambda \varphi, x \rangle n\lambda^{n-1}\, d\lambda \right\} -$$

$$- \int_a^b d\lambda \left(b^n \langle \varphi, x \rangle - \int_a^b \langle E_\lambda E_\mu \varphi, x \rangle n\mu^{n-1}\, d\mu \right) =$$

$$= b^{n+1} \langle \varphi, x \rangle - \int_a^b \langle E_\lambda \varphi, x \rangle (n+1) \lambda^n\, d\lambda .$$

Thus the required relation holds for all polynomials, and if $p(x)$ is any polynomial we have

$$| \langle \varphi, p(\tau) x \rangle | = \left| p(b) \langle \varphi, x \rangle - \int_a^b \langle E_\lambda \varphi, x \rangle p'(\lambda)\, dx \right| \leq$$

$$\leq p(b) | \langle \varphi, x \rangle | + \left(\int_a^b | \langle E_\lambda \varphi, x \rangle |^q d\lambda \right)^{1/q} \left(\int_a^b | p'(\lambda) |^p d\lambda ;\right.$$

hence we obtain

$$\| p(\tau) \| \leq (| p(b) |^p + [\mathrm{var}_p^p\, p(\lambda)]^{1/p}$$

and this proves our theorem as in the case $p = 1$ since the functionals

$$L_1(f) = \langle \varphi , f(\tau)\lambda \rangle$$

$$L_2(f) = f(b) \langle \varphi , x \rangle - \int_a^b \langle E_\lambda \varphi , x \rangle f(\lambda) \, d\lambda$$

coincide for polynomials.

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