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On wave solutions of the field equations of general relativity

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Fisica matematica. — *On wave solutions of the field equations of general relativity*. Nota (*) di KRISHNA B. LAL e SHAFIULLAH, presentata dal Socio C. CATTANEO.

RIASSUNTO. — Pandey e Rao hanno studiato in [1] l'equazione (1.1). Qui si studia l'equazione di campo della relatività generale di Einstein in uno spazio-tempo più generale, e si mostra ch'essa ammette per soluzioni delle onde piane e che il campo elettromagnetico risulta nullo.

1. INTRODUCTION

S. N. Pandey and J. K. Rao [1] (1) have obtained the solutions of Einstein and Maxwell's field equations in the presence of gravitational radiation, having the symmetry of the approximate field, for the space time:

$$ds^2 = dt^2 - dx^2 - (1 - A)\,dy^2 - (1 + A)\,dz^2 + 2\,B\,dy\,dz\,, \tag{1.1}$$

where A, B are functions of $t - x$. The above line element is always compatible with a null electromagnetic field.

Lal and Singh [2] have found out the plane wave solutions of various unified field equations with g_{ij} as non-symmetric tensor in the space-time (1.1). It is evident that gravitational waves are in the direction of negative x-axis.

In this paper the Authors propose to consider a more general space time which is given by

$$ds^2 = -(1 - A)\,dx^2 - (1 + A)\,dy^2 - C dz^2 + C dt^2 + 2\,B\,dx\,dy \tag{1.2}$$

where A, B are functions of $Z = z - t$ and $C \equiv C(z, t)$. (det. $(g) = mC^2$, where $m = 1 - A^2 - B^2$), and to obtain the plane wave solutions of the field equations of general relativity. Here plane waves are along the positive direction of the z-axis.

Einstein's gravitational field equations of general relativity are given by

$$R_{ij} = -8\,\pi T_{ij}\,, \qquad (i, j = 1, 2, 3, 4)\,, \tag{1.3}$$

where R_{ij} is the Ricci tensor and T_{ij} is the electromagnetic energy tensor given by

$$T_{ij} = \frac{1}{4} g_{ij} F^{lm} F_{lm} - F_{il} F_{jm} g^{lm}, \tag{1.4}$$

where F_{ij} is the skew symmetric electromagnetic field tensor satisfying the generalized Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = F_{ij,k} + F_{jk,i} + F_{k,ij} = 0\,, \tag{1.5}$$

$$F^{ij}_{;j} = 0\,. \tag{1.6}$$

(*) Pervenuta all'Accademia il 28 luglio 1976.

(1) Numbers in brackets refer to the references at the end of the paper.

Here a semicolon and a comma denote covariant and partial differentiation, with respect to the index occuring after them respectively.

2. Calculation of the Christoffel symbols and of the Ricci tensor

The non-vanishing components of the contravariant tensor g^{ij} in the metric (1.2) are

$$g^{11} = -\frac{1+A}{m} \quad , \quad g^{12} = -\frac{B}{m} = g^{21} , \qquad (2.1)$$

$$g^{22} = -\frac{1-A}{m} \quad , \quad g^{33} = -\frac{1}{C} = -g^{44} ;$$

and the non-vanishing Christoffel symbols of the second kind are

$$\left\{ {1 \atop 13} \right\} = -\frac{(1+A)\bar{A} + B\bar{B}}{2m} = -\left\{ {1 \atop 14} \right\} ,$$

$$\left\{ {1 \atop 23} \right\} = -\frac{(1+A)\bar{B} - B\bar{A}}{2m} = -\left\{ {1 \atop 24} \right\} ,$$

$$\left\{ {2 \atop 13} \right\} = -\frac{(1-A)\bar{B} + B\bar{A}}{2m} = -\left\{ {2 \atop 14} \right\} , \qquad (2.2)$$

$$\left\{ {2 \atop 23} \right\} = \frac{(1-A)\bar{A} - B\bar{B}}{2m} = -\left\{ {2 \atop 24} \right\} ,$$

$$\left\{ {3 \atop 11} \right\} = \frac{\bar{A}}{2C} = -\left\{ {3 \atop 22} \right\} = \left\{ {4 \atop 11} \right\} = -\left\{ {4 \atop 22} \right\} \quad , \quad \left\{ {3 \atop 12} \right\} = \frac{\bar{B}}{2C} = \left\{ {4 \atop 12} \right\} ,$$

$$\left\{ {3 \atop 33} \right\} = \frac{C_3}{2C} = \left\{ {3 \atop 44} \right\} = \left\{ {4 \atop 34} \right\} \quad , \quad \left\{ {3 \atop 34} \right\} = \frac{C_4}{2C} = \left\{ {4 \atop 33} \right\} = \left\{ {4 \atop 44} \right\} ,$$

where the indices $3 , 4$ attached to C denote its partial derivatives with respect to z and t respectively. A overhead bar denotes partial differentiation with respect to Z. The non-vanishing components of the curvature tensor

$R_{ijkl} (= -R_{jikl} = -R_{ijlk} = R_{klij})$ are given by

$$R_{1313} = -R_{1314} = R_{1414} = \frac{P}{2} ,$$

$$R_{1323} = -R_{1324} = R_{1424} = \frac{Q}{2} , \qquad (2.3)$$

$$R_{2324} = -R_{2323} = -R_{2424} = \frac{L}{2} ,$$

$$R_{3434} = -\frac{C_{33} - C_{44}}{2} + \frac{C_3^2 - C_4^2}{2C} = N ,$$

where

$$P = -\bar{\bar{A}} - \frac{2B\bar{A}\bar{B} + \bar{A}^2 + \bar{B}^2 + A(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{A}(C_3 - C_4)}{2C},$$

$$(2.4) \qquad Q = -\bar{\bar{B}} - \frac{2A\bar{A}\bar{B} - B(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{B}(C_3 - C_4)}{2C},$$

$$L = -\bar{\bar{A}} - \frac{2B\bar{A}\bar{B} - \bar{A}^2 - \bar{B}^2 + A(\bar{A}^2 - \bar{B}^2)}{2m} + \frac{\bar{A}(C_3 - C_4)}{2C}.$$

The non-vanishing components of the Ricci tensor obtained either from (2.3) on contraction with the help of (2.1) or by using (2.2) in the expression for the Ricci tensor

$$R_{ij} = -\begin{Bmatrix} a \\ ij \end{Bmatrix}_{,a} + \begin{Bmatrix} a \\ ia \end{Bmatrix}_{,j} - \begin{Bmatrix} b \\ ij \end{Bmatrix}\begin{Bmatrix} a \\ ba \end{Bmatrix} + \begin{Bmatrix} b \\ ia \end{Bmatrix}\begin{Bmatrix} a \\ bj \end{Bmatrix}$$

are as follows:

$$(2.5) \qquad R_{33} = M - \frac{N}{C}, \quad R_{44} = M + \frac{N}{C}, \quad R_{34} = -M,$$

where $M = \left[A(P+L) + 2BQ - \dfrac{\bar{A}^2 + \bar{B}^2}{m}\right] / 2m$.

3. THE ELECTROMAGNETIC FIELD

The components of the electromagnetic field F_{ij}, as obtained by Takeno [3], are given by

$$(3.1) \qquad F_{ij} = \begin{bmatrix} 0 & 0 & -\sigma & \sigma \\ 0 & 0 & \rho & -\rho \\ \sigma & -\rho & 0 & 0 \\ -\sigma & \rho & 0 & 0 \end{bmatrix},$$

where ρ and σ are arbitrary functions of Z only.

The contravariant components of the electromagnetic field tensor F^{ij} are therefore found to be

$$(3.2) \quad F^{ij} = \begin{bmatrix} 0 & 0 & \dfrac{B\rho - \sigma(1+A)}{mC} & \dfrac{B\rho - \sigma(1+A)}{mC} \\ 0 & 0 & \dfrac{\rho(1-A) - B\sigma}{mC} & \dfrac{\rho(1-A) - B\sigma}{mC} \\ -\dfrac{B\rho - \sigma(1+A)}{mC} & -\dfrac{B\sigma - \rho(1-A)}{mC} & 0 & 0 \\ -\dfrac{B\rho - \sigma(1+A)}{mC} & -\dfrac{\rho(1-A) - B\sigma}{mC} & 0 & 0 \end{bmatrix}.$$

Using (2.1), (3.1) and (3.2) in (1.4), the non-vanishing components of the electromagnetic energy tensor T_{ij} are given by

$$(3.3)\qquad T_{33} = -T_{34} = T_{44} = \frac{\sigma^2(1+A) - 2\sigma\rho B + \rho^2(1-A)}{m} .$$

Again, the components of the dual tensor F^*_{ij} of the electromagnetic field tensor F_{ij} are given by

$$F^*_{ij} = \tfrac{1}{2}\sqrt{-g}\,\delta_{ijlm}\,[4] ,$$

where δ_{ijlm} is the Levi-Civita symbol which is skew-symmetric in all four indices, the only non-vanishing components of δ_{ijlm} are those for which all four indices are different and they are equal to ± 1 according as (i, j, l, m) is an even or an odd permutation of (1, 2, 3, 4). The non-vanishing components of F^*_{ij} are given by

$$F^*_{23} = \sqrt{-g}\,F^{14} ,\quad F^*_{31} = \sqrt{-g}\,F^{24} ,\quad F^*_{14} = \sqrt{-g}\,F^{23} ,\quad F^*_{24} = \sqrt{-g}\,F^{31}\quad [4] .$$

Thus we have

$$(3.4)\quad F^*_{ij} = \begin{bmatrix} 0 & 0 & -\frac{\rho(1-A)-B\sigma}{\sqrt{m}} & \frac{\rho(1-A)-B\sigma}{\sqrt{m}} \\ 0 & 0 & \frac{B\rho-\sigma(1+A)}{\sqrt{m}} & -\frac{B\rho-\sigma(1+A)}{\sqrt{m}} \\ \frac{\rho(1-A)-B\sigma}{\sqrt{m}} & -\frac{B\rho-\sigma(1+A)}{\sqrt{m}} & 0 & 0 \\ -\frac{\rho(1-A)-B\sigma}{\sqrt{m}} & \frac{B\rho-\sigma(1+A)}{\sqrt{m}} & 0 & 0 \end{bmatrix}$$

and the contravariant components F^{*ij} are given by

$$(3.5)\qquad F^{*ij} = \begin{bmatrix} 0 & 0 & -\frac{\rho}{C\sqrt{m}} & -\frac{\rho}{C\sqrt{m}} \\ 0 & 0 & -\frac{\sigma}{C\sqrt{m}} & -\frac{\sigma}{C\sqrt{m}} \\ \frac{\rho}{C\sqrt{m}} & \frac{\sigma}{C\sqrt{m}} & 0 & 0 \\ \frac{\rho}{C\sqrt{m}} & \frac{\sigma}{C\sqrt{m}} & 0 & 0 \end{bmatrix} .$$

From (3.1), (3.2) and (3.4) it can be seen that $F_{ij} = F^{ij} = F^{*}_{ij} F^{ij} = 0$ showing that the electromagnetic field is null in the sense of Synge [5].

4. Solution of field equations (1.3), (1.5) and (1.6)

Substituting the values of R_{ij} and T_{ij} from (2.5) and (3.3) respectively into the field equation (1.3) we have

$$(4.1)\quad M - \frac{N}{C} = M + \frac{N}{C} = -M = -8\pi\left(\frac{\sigma^2(1+A) - 2\sigma\rho B + \rho^2(1-A)}{m}\right)$$

which are reduced to

$$(4.2)\qquad M = 8\pi\left(\frac{\sigma^2(1+A) - 2\sigma\rho B + \rho^2(1-A)}{m}\right)$$

and

$$(4.3)\qquad N = -\frac{(C_{33} - C_{44})}{2} + \frac{(C_3^2 - C_4^2)}{2C} = 0$$

which on integration gives

$$(4.4)\qquad \log C = f_1(z-t) + f_2(z+t)$$
$$\text{i.e.}\quad C = \exp\left(f_1(z-t) + f_2(z-t)\right).$$

The generalized Maxwell's field equations (1.5) and (1.6) are satisfied identically when F_{ij} and F^{ij} given by (3.1) and (3.2) are substituted in them.

Therefore the fundamental tensor g_{ij} satisfying (1.2) and the electromagnetic field tensor F_{ij} given by (3.1) constitute the plane wave solutions of the field equations (1.3), (1.5) and (1.6) under the conditions (4.2) and (4.4).

It is easy to observe that if we put

$$(4.5)\qquad C = 1 \quad , \quad B = 0$$

and use (2.2), (3.2) in (1.6), we obtain

$$\frac{\bar{A}}{1-A} + \frac{2\bar{\sigma}}{\sigma} = 0 \quad \text{and} \quad \frac{2\bar{\rho}}{\rho} - \frac{\bar{A}}{(1+A)} = 0$$

which on integration give

$$(4.6)\qquad a)\ \sigma = k\sqrt{1-A} = -F_{13} = F_{14},$$
$$b)\ \rho = k\sqrt{1+A} = F_{23} = -F_{24}$$

where k is a constant of integration.

Thus we get the solution as obtained by Pandey and Rao [1].

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