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On wave solutions of Einstein's field equations of general relativity containing zero-rest-mass scalar fields

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Fisica matematica. — *On wave solutions of Einstein's field equations of general relativity containing zero-rest-mass scalar fields.* Nota (*) di KRISHNA BEHARI LAL e ANIRUDH PRADHAN, presentata dal Socio C. CATTANEO.

RIASSUNTO. — Recentemente gli Autori hanno studiato le soluzioni ondose piane delle equazioni della relatività generale con campo elettromagnetico in uno spazio tempo generalizzato di Peres. In questa Nota essi ricavano le analoghe soluzioni, nella stessa metrica generalizzata, in presenza di un campo scalare di massa propria nulla.

1. INTRODUCTION

The study of relativistic field equations in the presence of scalar meson field has drawn the attention of many researchers. Brahmachary [1], Bergmann and Leipnik [2] have investigated the spherically symmetric fields associated with zero-rest-mass. The static solutions for axially and spherically symmetric fields have been investigated by Buchdahl [3] and he also studied the physical aspects of these solutions. Janis, Newman and Winicour [4], in an attempt to present an extension of Israel's [5] idea of singular event horizon, considered the spherically symmetric solutions of the field equations of general relativity containing zero-rest-mass meson fields. Penney [6] and Gautreau [7] have extended the study to the case of axially symmetric fields and have found that the scalar field obeys a flat-space Laplace equation such that a large class of solutions exist. Lal and Singh ([8], [9]) have obtained exact cylindrical wave solutions of Einstein's field equations of general relativity in presence of zero-rest-mass scalar fields and have further analyzed the non-singular nature of one of these solutions. Recently, Singh [10] and Patel [11] have investigated plane-symmetric solutions of the field equations corresponding to zero-mass scalar fields.

This paper is a continuation of [12] in which we have investigated the plane wave-like solutions of the field equations of general relativity containing electromagnetic fields in a generalized Peres space-time whose geometry is described by the line element

$$(1.1) \qquad ds^2 = -A\,dx^2 - B\,dy^2 - (1 - E)\,dz^2 - 2E\,dz\,dt + (1 + E)\,dt^2\,,$$

where A, B and E are functions of (x, y, Z); $Z \equiv (z - t)$. The object of this investigation is to find out the plane wave-like solutions (in the sense of Takeno [13]) of Einstein's field equations of general relativity in presence

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of a zero-rest-mass scalar field. The equations under consideration are given by

$$G_{ij} \equiv R_{ij} - \tfrac{1}{2} g_{ij} R = -8 \pi T_{ij} \tag{1.2}$$

$$g^{ij} \varphi_{;ij} = 0 , \tag{1.3}$$

where φ is a scalar field having zero-rest-mass and T_{ij}, the energy momentum tensor of this field, is defined by

$$T_{ij} = \varphi_{,i} \varphi_{,j} - \tfrac{1}{2} g_{ij} (\varphi_{,l} \varphi_{,m} g^{lm}). \tag{1.4}$$

Here and elsewhere a semicolon indicates covariant differentiation and a comma followed by an index i denotes partial differentiation with respect to x^i.

2. Calculation of G_{ij} and T_{ij}

The non-vanishing components of the Ricci tensor as given in [12] are:

$$\begin{aligned} R_{11} &= L/B \quad , \quad R_{13} = -R_{14} = -N/B \quad , \quad R_{22} = L/A , \\ R_{23} &= -R_{24} = M/A \quad , \quad R_{33} = -R_{34} = R_{44} = \alpha/A + \eta/B . \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} 2L &= A_{yy} + B_{xx} - \tfrac{1}{2}(A_y^2 + A_x B_x)/A - \tfrac{1}{2}(B_x^2 + A_y B_y)/B , \\ 2M &= \bar{A}_y - \tfrac{1}{2} A_y (\bar{A}/A + \bar{B}/B) , \\ 2N &= -\bar{B}_x + \tfrac{1}{2} B_x (\bar{A}/A + \bar{B}/B) , \\ 2\alpha &= \bar{\bar{A}} - E_{xx} - \tfrac{1}{2}(\bar{A}^2 - A_x E_x)/A - \tfrac{1}{2} A_y E_y/B , \\ 2\eta &= \bar{\bar{B}} - E_{yy} - \tfrac{1}{2}(\bar{B}^2 - B_y E_y)/B - \tfrac{1}{2} B_x E_x/A . \end{aligned} \tag{2.2}$$

From here onwards, the lower suffixes x and y attached with any function indicate partial differentiation with respect to x and y respectively and a bar over any function denotes partial differentiation with respect to Z.

By a straightforward calculation we find the non-vanishing components of G_{ij} and T_{ij} for the metric (1.1) to have the following values:

$$\begin{aligned} & G_{13} = -G_{14} = -N/B \quad , \quad G_{23} = -G_{24} = M/A , \\ & G_{33} + L/AB = -G_{34} = G_{44} - L/AB = (B\alpha + A\eta + EL)/AB , \end{aligned} \tag{2.3}$$

and

$$(2.4)\qquad T_{11} = \tfrac{1}{2}\,\varphi_{,1}^{2} - (A/2\,B)\,\varphi_{,2}^{2} - AE\,\varphi_{,3}\,\varphi_{,4} - \tfrac{1}{2}\,A\,(\varphi_{,3}^{2} - \varphi_{,4}^{2}) - \tfrac{1}{2}\,AE\,(\varphi_{,3}^{2} + \varphi_{,4}^{2})\,,$$

$$T_{22} = \tfrac{1}{2}\,\varphi_{,2}^{2} - (B/2\,A)\,\varphi_{,1}^{2} - BE\,\varphi_{,3}\,\varphi_{,4} - \tfrac{1}{2}\,B\,(\varphi_{,3}^{2} - \varphi_{,4}^{2}) - \tfrac{1}{2}\,BE\,(\varphi_{,3}^{2} + \varphi_{,4}^{2})\,,$$

$$T_{12} = \varphi_{,1}\,\varphi_{,2}\quad,\quad T_{13} = \varphi_{,1}\,\varphi_{,3}\quad,\quad T_{14} = \varphi_{,1}\,\varphi_{,4}\,,$$

$$T_{23} = \varphi_{,2}\,\varphi_{,3}\quad,\quad T_{24} = \varphi_{,2}\,\varphi_{,4}\,,$$

$$T_{33} = \tfrac{1}{2}\,(1 - E)\,(\varphi_{,1}^{2}/A + \varphi_{,2}^{2}/B) + \tfrac{1}{2}\,(1 + E^{2})\,(\varphi_{,3}^{2} + \varphi_{,4}^{2}) - E\,(1 - E)\,\varphi_{,3}\,\varphi_{,4} - E\varphi_{,4}^{2}\,,$$

$$T_{34} = -\tfrac{1}{2}\,E\,(\varphi_{,1}^{2}/A + \varphi_{,2}^{2}/B) + (1 - E^{2})\,\varphi_{,3}\,\varphi_{,4} - \tfrac{1}{2}\,E\,(\varphi_{,3}^{2} - \varphi_{,4}^{2}) - \tfrac{1}{2}\,E^{2}\,(\varphi_{,3}^{2} + \varphi_{,4}^{2})\,,$$

$$T_{44} = \tfrac{1}{2}\,(1 + E)\,(\varphi_{,1}^{2}/A + \varphi_{,2}^{2}/B) + E\,(1 + E)\,\varphi_{,3}\,\varphi_{,4} + \tfrac{1}{2}\,(1 + E^{2})\cdot(\varphi_{,3}^{2} + \varphi_{,4}^{2}) + E\varphi_{,3}^{2}\,.$$

3. SOLUTIONS OF EQUATIONS (1.2) AND (1.3)

Using (1.1) in the field equations (1.2) and (1.3) we get

$$(3.1)\qquad \varphi_{,1}^{2} - (A/B)\,\varphi_{,2}^{2} - A\,(\varphi_{,3}^{2} - \varphi_{,4}^{2}) - AE\,(\varphi_{,3} + \varphi_{,4})^{2} = 0\,,$$

$$(3.2)\qquad \varphi_{,2}^{2} - (B/A)\,\varphi_{,1}^{2} - B\,(\varphi_{,3}^{2} - \varphi_{,4}^{2}) - BE\,(\varphi_{,3}^{2} + \varphi_{,4}^{2}) = 0\,,$$

$$(3.3)\qquad \varphi_{,1}\,\varphi_{,2} = 0\,,$$

$$(3.4)\qquad N/B = 8\,\pi\varphi_{,1}\,\varphi_{,3}\,,$$

$$(3.5)\qquad N/B = -8\,\pi\varphi_{,1}\,\varphi_{,4}\,,$$

$$(3.6)\qquad M/A = -8\,\pi\varphi_{,2}\,\varphi_{,3}\,,$$

$$(3.7)\qquad M/A = 8\,\pi\varphi_{,2}\,\varphi_{,4}\,,$$

$$(3.8)\qquad B\alpha + A\eta + EL - L = -8\,\pi\,AB\,[\tfrac{1}{2}\,(1 - E)\,(\varphi_{,1}^{2}/A + \varphi_{,2}^{2}/B) + \tfrac{1}{2}\,(1 + E^{2})\,(\varphi_{,3}^{2} + \varphi_{,4}^{2}) - E\,(1 - E)\,\varphi_{,3}\,\varphi_{,4} - E\varphi_{,4}^{2}]\,,$$

$$\text{(3.9)} \quad B\alpha + A\eta + EL = 8\pi AB\,[-\tfrac{1}{2}E(\varphi_{,1}^2/A + \varphi_{,2}^2/B) + (1 - E^2)\varphi_{,3}\varphi_{,4} - \tfrac{1}{2}E(\varphi_{,3}^2 - \varphi_{,4}^2) - \tfrac{1}{2}E^2(\varphi_{,3}^2 + \varphi_{,4}^2)]\,,$$

$$\text{(3.10)} \quad B\alpha + A\eta + EL + L = -8\pi AB\,[\tfrac{1}{2}(1 + E)(\varphi_{,1}^2/A + \varphi_{,2}^2/B) + E(1 + E)\varphi_{,3}\varphi_{,4} + \tfrac{1}{2}(1 + E^2)(\varphi_{,3}^2 + \varphi_{,4}^2) + E\varphi_{,3}^2]\,,$$

$$\text{(3.11)} \quad \frac{1}{A}\{\varphi_{,11} - (A_x/2A)\varphi_{,1} + (A_y/2B)\varphi_{,2} + \tfrac{1}{2}\bar{A}\varphi_{,3} + \tfrac{1}{2}\bar{A}\varphi_{,4}\} + \frac{1}{B}\{\varphi_{,22} + (B_x/2A)\varphi_{,1} - (B_y/2B)\varphi_{,2} + \tfrac{1}{2}\bar{B}\varphi_{,3} + \tfrac{1}{2}\bar{B}\varphi_{,4}\} - {}$$
$$+ \varphi_{,33} + \varphi_{,44} - E(\varphi_{,33} + 2\varphi_{,34} + \varphi_{,44}) + 2E\{(E_x/A)\varphi_{,1} + (E_y/B)\varphi_{,2} - \bar{E}\varphi_{,3} + \bar{E}\varphi_{,4}\} = 0\,.$$

Equations (3.4), (3.5) or equations (3.6), (3.7) imply

$$\text{(3.12)} \quad \varphi_{,3} + \varphi_{,4} = 0\,.$$

The three possibilities, namely,

(i) $\varphi_{,1} = 0 \quad , \quad \varphi_{,2} \neq 0\,,$

(ii) $\varphi_{,1} \neq 0 \quad , \quad \varphi_{,2} = 0\,,$

(iii) $\varphi_{,1} = 0 \quad , \quad \varphi_{,2} = 0\,,$

which arise from (3.3), when considered with (3.12), (3.1) and (3.2) give

$$\text{(3.13)} \quad \varphi_{,1} = 0 \quad , \quad \varphi_{,2} = 0 \quad \text{and} \quad \varphi_{,3} + \varphi_{,4} = 0\,,$$

which shows that φ is function of Z alone. Consequently equations (3.1), (3.2), (3.3) and (3.11) are identically satisfied and the remaining seven equations from (3.4) to (3.10) reduce to

$$\text{(3.14)} \quad \bar{B}_x - \tfrac{1}{2}B_x(\bar{A}/A + \bar{B}/B) = 0\,,$$

$$\text{(3.15)} \quad \bar{A}_y - \tfrac{1}{2}A_y(\bar{A}/A + \bar{B}/B) = 0\,,$$

$$\text{(3.16)} \quad (B_x/\sqrt{AB})_x + (A_y/\sqrt{AB})_y = 0\,,$$

$$\text{(3.17)} \quad \frac{1}{A}\{\bar{\bar{A}} - E_{xx} - \tfrac{1}{2}(\bar{A}^2 - A_x E_x)/A - \tfrac{1}{2}A_y E_y/B\} + \frac{1}{B}\{\bar{\bar{B}} - E_{yy} - \tfrac{1}{2}(\bar{B}^2 - B_y E_y)/B - \tfrac{1}{2}B_x E_x/A\} = -16\pi\bar{\varphi}^2.$$

Equations (3.14) and (3.15), after integration, give

$$B_x/\sqrt{AB} = k_1 , \tag{3.18}$$

$$A_y/\sqrt{AB} = k_2 , \tag{3.19}$$

respectively, where in general k_1 and k_2 are functions of (x , y). Equations (3.16)-(3.19) are mathematically complicated and it is not possible to get exact solutions. Let us consider a relation

$$A = Bf , \qquad (f = f(Z)) . \tag{3.20}$$

From (3.18) and (3.19) we get

$$B_x A_y = AB\, k_1 k_2 . \tag{3.21}$$

With the help of (3.20), equation (3.21) reduces to

$$pq = \psi , \qquad (p = B_x , q = B_y , \psi = k_1 k_2 B^2) . \tag{3.22}$$

Foregoing equation is a standard form of differential equation which can be solved by using Charpit's method [14].

For finding the value of E, we use (3.20) in (3.17), which on simplification gives

$$E_{xx}/f + E_{yy} = 2\bar{\bar{B}} - (\bar{B})^2/B + \bar{B}\bar{f}/f - \tfrac{1}{2}(\bar{f}/f)^2 B + 16\pi\bar{\varphi}^2 . \tag{3.23}$$

Thus the plane wave-like solutions of the field equations (1.2) and (1.3) are composed of g_{ij} given by (1.1) satisfying conditions (3.16), (3.20) and (3.23).

It may be mentioned that if we take $k_1 = k_2 = 0$, equations (3.18) and (3.19) give $B_x = A_y = 0$ so that $A = A(x , Z)$ and $B = B(y , Z)$. Thus the g_{ij} given by (1.1) with $A = A(x , Z)$, $B = B(y , Z)$, $E = E(x , y , Z)$ and $\varphi = \varphi(Z)$ represent the plane wave-like solutions of the field equations (1.2) and (1.3) provided the condition

$$\frac{1}{A}\{\bar{\bar{A}} - E_{xx} - (\bar{A}^2 - A_x E_x)/2A\} + \frac{1}{B}\{\bar{\bar{B}} - E_{yy} - (\bar{B}^2 - B_y E_y)/2B\} = 16\pi\bar{\varphi}^2 , \tag{3.24}$$

holds.

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