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Fisica matematica. — *On the evolution system for a relativistic inviscid fluid with heat conduction* (*). Nota di ANTONIO GRECO e SEBASTIANO GIAMBÒ, presentata (**) dal Socio C. CATTANEO.

RIASSUNTO. — Si stabilisce che il tensore energia–impulso di Eckart–Pham è, in un certo senso, una prima approssimazione del tensore di Landau–Carini. Viene poi considerato il sistema differenziale di evoluzione del fluido, associando allo schema di Landau una conveniente equazione per la conduzione del calore. Si studiano le principali conseguenze dell'associazione considerata riguardo alla propagazione ondosa nel caso di un fluido perfetto politropico.

1. INTRODUCTION

As is well known, the determination of the energy-momentum tensor for a relativistic inviscid heat-conducting fluid is still an open question.

Here we recall only the two most quoted formulations: the first one, given originally by C. Eckart [1], reconsidered and modified, among others by Pham Mau Quan [2], includes in the expression of the energy-momentum tensor some symetrized terms directly involving the heat-flux vector: the second one, given by L. D. Landau [3] on the basis of physical intuitions, preserves the same structure for the energy-momentum tensor as that for an inviscid non heat-conducting fluid.

Recently, G. Carini [4] has confirmed the Landau formulation deducing the energy-momentum tensor on the basis of the dynamics of a particle with variable rest-mass and making use of an appropriate definition of the rest frame for the fluid particles. This frame, as already noticed by Landau [3], is the inertial system in which the total momentum, i.e. inertial momentum plus thermal momentum, vanishes.

In this paper, after having stated the notations at the end of this section, in section 2 we briefly recall the two above schemes and prove that it is possible to obtain one of them from the other one, if this latter is expressed in terms of an asymptotic expansion starting from a thermically unperturbed state. In this manner the Eckart-Pham tensor is obtained if we truncate the expansion just at the first perturbation term. In section 3 the fundamental sistem given in [4], together with a suitable relativistic generalization of Cattaneo's equation for the heat conduction, will be recalled and transformed in an equivalent form. In sections 4 and 5 infinitesimal discontinuities and characteristic equations will be discussed. In section 6, in the case of a perfect polytropic fluid, the velocities of propagation, possible exceptionality of

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corresponding waves, associated discontinuities, rays and reasonable thermodynamical inequalities, ensuring the spatial orientation of the waves, will be considered. The main result is the determination of a new wave, directly depending on the thermal conduction, and a different behaviour of the discontinuities across the usual hydrodynamical waves which are still present.

These results are different from those obtained by Boillat [5] on account of the different energy-momentum tensor used, and from those of Mahjoub [6] on account of the different equation of thermal conduction.

NOTATIONS. The space-time is a four dimensional manifold V^4, whose normal hyperbolic metric ds^2, with signature $+---$, is expressible in local coordinates in the usual form: $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$; the metric tensor is assumed to be given of class C^1, piecewise C^2; the four velocity is defined as $u^\alpha = dx^\alpha/ds$ which implies its unitary character: $u^\alpha u_\alpha = 1$; ∇_α is the operator of covariant differentiation with respect to the given metric.

2. THE TWO SCHEMES

In the Landau-Carini scheme the energy-momentum tensor for an inviscid heat-conducting fluid is formally the same as that for a non heat conducting one:

$$T^{\alpha\beta} = c^2\, rfu^\alpha\, u^\beta - pg^{\alpha\beta} \tag{1}$$

where r is the proper material density (corresponding to the particles number), $f = 1 + i/c^2$ the index of the fluid, i the specific enthalpy (the so called heat function) which in this case accounts also for the heat amount, p the hydrodynamical pressure and u^α the unitary four-velocity.

The heat flux vector Q^α consists of a convective part and a conductive one, orthogonal to the four-velocity:

$$Q^\alpha = q^*\, u^\alpha + q^\alpha \quad , \quad q^\alpha\, u_\alpha = 0 \tag{2}$$

and satisfies a separated conservation equation.

On the other hand in the Eckart-Pham scheme the energy-momentum tensor is given by

$$\tilde{T}^{\alpha\beta} = c^2\, \tilde{r}\tilde{f}\tilde{u}^\alpha\, \tilde{u}^\beta - \tilde{p}g^{\alpha\beta} + \tilde{q}^\alpha\, \tilde{u}^\beta + \tilde{q}^\beta\, \tilde{u}^\alpha \tag{1'}$$

with the supplementary hypothesis

$$\tilde{q}^\alpha\, \tilde{u}_\alpha = 0. \tag{2'}$$

To explain the different physical meaning of the two four-velocities u^α and $\tilde{u}^\alpha$ it is more convenient to work in the context of special relativity. Here we have

$$u^\alpha = \left(\alpha\, , \frac{\alpha}{c}\, \boldsymbol{v}\right) \qquad \alpha = (1 - \beta^2)^{-\frac{1}{2}} \qquad \beta^2 = \frac{v^2}{c^2}$$

$$\tilde{u}^\alpha = \left(\tilde{\alpha}\, , \frac{\tilde{\alpha}}{c}\, \tilde{\boldsymbol{v}}\right) \qquad \tilde{\alpha} = (1 - \tilde{\beta}^2)^{-\frac{1}{2}} \qquad \tilde{\beta}^2 = \frac{\tilde{v}^2}{c^2}$$

where $\boldsymbol{v}$ is the ordinary velocity of the inertial frame in which the following relation holds

$$c^2 r\boldsymbol{v} + \boldsymbol{q} = 0 , \tag{3}$$

$\boldsymbol{q}$ being the ordinary heat-flux vector, $\tilde{\boldsymbol{v}}$ the velocity of the rest frame for the fluid particle which corresponds to the material density r alone, as usually considered in absence of the heat flux. So, the most significant difference between the two schemes consists in the definition of the rest frame of a fluid particle. The former is defined, in agreement with the equation (3), as the inertial system in which the total momentum (inertial+thermal) vanishes. The latter does not take into account a thermal momentum. As pointed-out by Carini [4], the appropriate definition of the rest frame and the systematic use of the relativistic dynamics of a particle with variable rest-mass, leads to the form (1) for the energy-momentum tensor and to the consequent conservation equations, in agreement with the scheme previously proposed by Landau [3].

Now, to prove that the Eckart-Pham tensor is an approximation, in a definite sense, of that given by Landau-Carini, we first consider an inviscid fluid in adiabatic conditions. We call this state, which will be denoted by the suffix $_0$, thermically unperturbed state. As is well known, the energy-momentum tensor is

$$T_0^{\alpha\beta} = c^2 r_0 f_0 u_0^\alpha u_0^\beta - p_0 g^{\alpha\beta}.$$

Now we suppose that a thermal process, with convective and conductive effects, occurs. In this state the energy-momentum tensor has the form (1).

Moreover, the above meaning of $\tilde{\boldsymbol{v}}$ allows: $r_0 = \tilde{r} = r$, $f_0 = \tilde{f} \neq f$, $p_0 = \tilde{p} = p$ and $u_0^\alpha = \tilde{u}^\alpha \neq u^\alpha$. At this point, as the thermal flux implies a change in the rest frame, it is quite natural to give the perturbed quantities f, u^α as asymptotic series in the parameter $1/c^2$ starting from the unperturbed ones f_0 and u_0^α:

$$f = f_0 + \frac{1}{c^2} f_1 + \frac{1}{c^4} f_2 + \cdots$$

$$u^\alpha = u_0^\alpha + \frac{1}{c^2} u_1^\alpha + \frac{1}{c^4} u_2^\alpha + \cdots .$$

By substituting in (1) f and u^α with their asymptotic expansions we obtain

$$T^{\alpha\beta} = T_0^{\alpha\beta} + \frac{1}{c^2} T_1^{\alpha\beta} + \cdots \tag{4}$$

with

$$T_1^{\alpha\beta} = u_0^\alpha \tilde{q}^\beta + u_0^\beta \tilde{q}^\alpha$$

where

$$\tilde{q}^\alpha = c^2 r \left(f_0 u_1^\alpha + \frac{1}{2} f_1 u_0^\alpha \right) .$$

Therefore, if we truncate the expansion (4) at the first perturbation term-with the above positions, we obtain the Eckart-Pham energy-momentum tensor.

Remark 1. In our discussion it is apparent that in the unperturbed state the heat-flux is identically zero: $q_0^\alpha = 0$, so the condition $u_0^\alpha q_{0\alpha} = 0$ is trivially satisfied. On the other hand the equation (2′) cannot be verified because the orthogonality condition is given by the equation (2_2).

Remark 2. We have written $\tilde{f} = f_0$. This is confirmed by the fact that Pham considers the thermodynamical relation $c^2 \, d\tilde{f} = (1/\tilde{r}) \, d\tilde{p} + \tilde{T} \, d\tilde{S}$ along the stream lines $c^2 \tilde{u}^\alpha \partial_\alpha \tilde{f} = (1/\tilde{r}) \tilde{u}^\alpha \partial_\alpha \tilde{p} + \tilde{T} \tilde{u}^\alpha \partial_\alpha \tilde{S}$. As $\tilde{u}^\alpha = u_0^\alpha$, this is correct only if $\tilde{f} = f_0$.

Remark 3. The result obtained here is still consistent if we consider, as customary, the asymptotic expansion in terms of an arbitrary parameter $1/\omega \ (\omega \gg 1)$. But if we choose $\omega = c^2$, the truncation at the first perturbation term agrees with the usual relativistic approximations in terms of the powers of $1/c^2$.

Comment. The first perturbation term resorts from a perturbation in the index of the fluid and a perturbation in the four-velocity. This is consistent with our hypotheses in which the heat flux does not change the number of particles or the hydrodynamical pressure. On the contrary, as the total momentum is modified, this influences from one hand the four-velocity, from the other hand the energy content (i.e. c^2 times the material content) of a fluid particle, which is reflected in the variation of the index of the fluid, which is essentially the heat function i.

In conclusion we observe that, as it is well known, the two schemes coincide in the classical limit (i.e. for $c \to \infty$). In the Landau–Carini approach u^α is connected with the energy transport, while in the Eckart-Pham approach it is connected with the particles transport. So we can reverse the procedure to obtain the Landau-Carini energy-momentum tensor as an approximation of the Eckart-Pham one. To do this it is necessary to give f and u^α in terms $\tilde{f} = \tilde{f}_0$, $\tilde{u}^\alpha = u_0^\alpha$ and $\tilde{q}^\alpha = \frac{1}{c^2} q_1^\alpha + \cdots$. We can choose $f = \tilde{f} = f_0$, $u^\alpha = u_0^\alpha + \frac{1}{c^2 r f} q_1^\alpha$.

But in the former procedure we can preserve the physical meaning of the quantities involved in the Eckart-Pham tensor as given by the Authors and we must reject the equation (2′) only. In the latter procedure we must reject the equation (2_2), we need not a separate conservation equation for the thermal energy, the thermodynamical differential equation is not rigorously used and the heat-flux vector is not completely free. For these reasons the Landau-Carini formulation seems to be more convenient and will be adopted in the following.

3. FIELD EQUATIONS

The field equations are the equation of conservation for the energy-momentum tensor

$$(6)\qquad \nabla_\alpha T^{\alpha\beta} = 0 \qquad T^{\alpha\beta} = c^2 rfu^\alpha u^\beta - pg^{\alpha\beta} ;$$

the conservation equation for the proper total material density, constituted by a pure material part and a part which comes from the thermal energy, on the basis of the Einstein principle of the inertia of the energy:

$$(7)\qquad \nabla_\alpha (\mu u^\alpha) = 0 \quad , \quad \mu = r + \frac{q^*}{c^2} ;$$

the conservation equation for the thermal energy:

$$(8)\qquad \nabla_\alpha (q^* u^\alpha + q^\alpha) = 0 \quad , \quad q^\alpha u_\alpha = 0 .$$

Moreover the relation

$$(9)\qquad c^2 \, \mathrm{d}f = \frac{1}{r} \, \mathrm{d}p + T \, \mathrm{d}S$$

which comes from the thermodynamics principles will be considered. Here T is the proper absolute temperature and S the specific proper entropy.

In the following we will assume p and S as the thermodynamical indipendent variables.

Finally we assume that the following heat equation, which is of the type recently suggested by Boillat [5], holds:

$$(10)\qquad q_\beta = k\gamma_\beta^\alpha \partial_\alpha T - kTu^\alpha \nabla_\alpha u_\beta - \chi u^\alpha (\nabla_\alpha q_\beta - \nabla_\beta q_\alpha) ; \qquad \partial_\alpha \equiv \partial/\partial x^\alpha .$$

The positive coefficients k and χ may depend on p and S, $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$.

This equation, neglecting the Eckart's term: $kTu^\alpha \nabla_\alpha u_\beta$, differs from the Kranys equation [6] in the last term of the right-hand side. The yielded modification, on one hand leads to the obvious advantage that the orthogonality condition (8_2) is automatically satisfied; on the other hand, as shown by Boillat, implies the reasonable fact that the heat flux, in some approximate sense, depends, among other things, also on the gradient of the substantial derivative of the temperature.

Now, from eq. (6), contracting by u_β, we obtain the continuity equation:

$$(11)\qquad \nabla_\alpha (c^2 rfu^\alpha) - u^\alpha \partial_\alpha p = 0 ,$$

which, by eqs. (7), (8) and taking into account the relation (9), can be written

$$(12)\qquad rTu^\alpha \partial_\alpha S + f \nabla_\alpha q^\alpha = 0 .$$

Finally, from eq. (6), according to eq. (11), we deduce the differential system of the stream lines:

$$c^2 rfu^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta} \partial_\alpha p = 0 . \tag{13}$$

Our main system is constituted by the eleven (scalar) equations (13), (7), (8), (12) and (10) in the eleven unknowns $u^\alpha, q^\alpha, q^*, p$ and S. Moreover the written equations are completely compatible with the two algebraic relations $u^\alpha u_\alpha = 1$, $u^\alpha q_\alpha = 0$.

4. Discontinuities

We now suppose that the field variables $u^\alpha, q^\alpha, q^*, p$ and S are of class C^0, piecewise C^1; that the discontinuities of their first order derivatives can take place across an hypersurface Σ of local equation $\varphi(x^\alpha) = 0$, φ of class C^2, that those discontinuities are well determined as difference of the limiting values of the derivativies of $u^\alpha, q^\alpha, q^*, p$ and S obtained approaching the same point of Σ by the two sides in which Σ divides V^4; that those limiting values are tensor functions defined on Σ and that the above derivatives are uniformly convergent to these functions when one tends to the points of Σ on either sides. In these hypotheses [7], introducing the operator of the infinitesimal discontinuity δ, we study in which conditions the tensor distributions δu^α, δq^α, δq^*, δp and δS, supported with regularity by Σ, are not simultaneously zero. At the same time we obtain the differential equation (i.e. the characteristic equation) which must be satisfied by the functions φ.

To this end it is sufficient to make the replacement $\nabla_\alpha, \partial_\alpha \to \varphi_\alpha \delta_\alpha$, $\varphi_\alpha \equiv \partial_\alpha \varphi$, in the above differential equation. So from eqs. (13), (7), (8), (12) and (10) we have respectively

$$c^2 rf U \, \delta u^\alpha - \gamma^{\alpha\beta} \varphi_\beta \, \delta p = 0 \tag{14}$$

$$\mu\varphi_\alpha \, \delta u^\alpha + U \left(\delta \frac{q^*}{c^2} + r'_p \, \delta p + r'_s \, \delta S \right) = 0 \tag{15}$$

$$q^* \varphi_\alpha \, \delta u^\alpha + \varphi_\alpha \, \delta q^\alpha + U \delta q^* = 0 \tag{16}$$

$$f \varphi_\alpha \, \delta q^\alpha + r T U \, \delta S = 0 \tag{17}$$

$$k T U \, \delta u^\alpha + \chi U \, \delta q^\alpha - \chi \varphi^\alpha u_\beta \, \delta q^\beta - k \gamma^{\alpha\beta} \varphi_\beta (T'_p \, \delta p + T'_S \, \delta S) = 0 \tag{18}$$

where $U \equiv u^\alpha \varphi_\alpha$ and the prime denotes partial differentiation with respect to subscripted variable.

Moreover, from the equation (8_2) and the unitary character of u_α we have

$$u_\alpha \, \delta u^\alpha = 0 \quad , \quad u_\alpha \, \delta q^\alpha + q_\alpha \, \delta u^\alpha = 0 . \tag{19}$$

5. WAVES

From the above equations, we first have the solution $U = 0$ which represents a wave moving with the fluid. For the corresponding discontinuities we find

$$\varphi_\alpha \, \delta u^\alpha = \varphi_\alpha \, \delta q^\alpha = \delta p = 0 \quad , \quad \chi G u_\alpha \, \delta q^\alpha + k C T'_S \, \delta S = 0$$

which come from eqs. (14), (15), (16) and (18) respectively. We have put $\varphi^\alpha \varphi_\alpha = G$ and $\gamma^{\alpha\beta} \varphi_\alpha \varphi_\beta = C$. As $U = 0$ implies $\varphi_\alpha \, \delta u^\alpha = 0$ it is an exceptional wave.

Taking into account eq. (19), we see that 5 degrees of freedom are left: one for δq^* and four between δu^α and δq^α; we have 5 indipendent eigenvectors corresponding to $U = 0$ in the space of the field variables.

From now on we suppose $U \neq 0$. The eq. (14) multiplied by φ_α gives

$$c^2 r f U \, \varphi_\alpha \, \delta u^\alpha - C \delta p = 0 \,, \tag{20}$$

and multiplied by q_α, taking account eq. (19_2):

$$c^2 r f U \, u_\alpha \, \delta q^\alpha + Q \delta p = 0 \;, \qquad Q \equiv q^\alpha \varphi_\alpha \,. \tag{21}$$

From eqs. (15), (16) and (17), by eliminating δq^* and $\varphi_\alpha \, \delta q^\alpha$, we have

$$c^2 r f \, \varphi_\alpha \, \delta u^\alpha + c^2 f r'_p U \, \delta p + U (r T + c^2 f r'_S) \, \delta S = 0$$

which, by virtue of relation (9), can be written as

$$c^2 r f \, \varphi_\alpha \, \delta u^\alpha + c^2 f r'_p U \, \delta p + (c^2 r f)'_S \, U \, \delta S = 0 \,. \tag{22}$$

Finally, from eq. (18), contracting by φ_α and utilizing eq. (17), we find

$$k f T U \, \varphi_\alpha \, \delta u^\alpha - \chi f G u_\alpha \, \delta q^\alpha - k f T'_p C \, \delta p - H \, \delta S = 0 \tag{23}$$

where

$$H = \chi r T U^2 + f T'_S C \,. \tag{24}$$

Eqs. (20)–(23) give a linear homogeneous system in the four scalar distributions $\varphi_\alpha \, \delta u^\alpha$, $u_\alpha \, \delta q^\alpha$, δp and δS. It follows that the just above mentioned distributions can be different from zero only if the determinant

$$\begin{vmatrix} c^2 r f U & 0 & -C & 0 \\ 0 & c^2 r f U & Q & 0 \\ c^2 r f & 0 & c^2 f r'_p U & (c^2 r f)'_S U \\ k f T U & -\chi f G & -f k T'_p C & -H \end{vmatrix}$$

vanishes.

Therefore, neglecting an inessential factor, we find for the hydrodynamical waves, the equation

$$P \equiv c^2 rf H (c^2 fr'_p U^2 + C) + (c^2 fr)'_S \{kf (T - c^2 rf T'_p) U^2 + \chi f UQG\} = 0 .$$

This provides in general four velocities and correspondently four linearly indipendent eigenvectors in the space of the field variables. It follows that if the conditions ensuring the reality of the velocities coming from $P = 0$ are given, the system is hyperbolic (not strictly). In fact all velocities (eigenvalues) are real, and there is a complete set of eigenvectors in the space of field variables: 9 indipendent eigenvectors (5 from $U = 0$, 4 from $P = 0$), for the 9 indipendent field variables $u^\alpha, q^\alpha, q^*, p$ and S.

This is the case, which will be considered in the following, of a perfect polytropic fluid.

6. Perfect Polytropic fluid

We now suppose that the termodynamical relation

$$(c^2 rf)'_S = 0$$

holds. This is equivalent to suppose

$$\rho = \rho(p) \quad , \quad \rho = T^{\alpha\beta} u_\alpha u_\beta ,$$

which includes the perfect polytropic fluid as a particular case.

In these conditions $P = 0$ factorises as

$$c^2 fr'_p U^2 + C = 0 , \tag{25}$$

$$\chi rTU^2 + kf T'_S C = 0 . \tag{26}$$

The first equation corresponds to the usual hydrodynamical waves, well known also in absence of thermal conduction. Their speed is given by

$$\frac{V_h^2}{c^2} = \frac{1}{c^2 fr'_p} \tag{27}$$

and the condition $c^2 fr'_p \geq 1$, already given, [7], ensures their spatial orientation. In general they are not exceptional, except in the case of the incompressible fluid [8]. The associated discontinuities, taking into account the relation $c^2 rf T'_p - T = 0$ which follows from $(c^2 rf)'_S = 0$, can be written in terms of δp as

$$\begin{cases} c^2 rf U \,\delta u^\alpha = \gamma^{\alpha\beta} \varphi_\beta \,\delta p \\ c^2 rf U^2 H \,\delta q^\alpha = (kf T'_S G \gamma^{\alpha\beta} \varphi_\beta - H\varphi^\alpha) Q \delta p \\ c^2 rf U^2 H \,\delta q^* = (\chi r \, TUQG - q^* HC) \,\delta p \\ c^2 rf U H \,\delta S = \chi QG \,\delta p . \end{cases} \tag{28}$$

We see that, unlike the case in which the thermal conduction is not taken into account, the entropy also may be discontinuous across these waves.

The waves given by the eq. (26) are a direct consequence of the supposed thermal conduction. Their spatial orientation follows by the thermodynamical inequalities

$$\chi r \mathrm{T} \geq kf \mathrm{T}'_{\mathrm{S}} \quad , \quad \mathrm{T}'_{\mathrm{S}} \geq 0 \tag{29}$$

which are supposed to hold. For their speed we find

$$\frac{\mathrm{V}_t^2}{c^2} = \frac{kf\mathrm{T}'_{\mathrm{S}}}{\chi r \mathrm{T}} , \tag{30}$$

and the discontinuities in terms of $\delta\mathrm{S}$ only are given by

$$\delta u^\alpha = \delta p = 0 \quad , \quad \chi \mathrm{U} \delta q^\alpha = k \mathrm{T}'_{\mathrm{S}} \gamma^{\alpha\beta} \varphi_\beta \, \delta \mathrm{S} \quad , \quad \chi \mathrm{U}^2 \, \delta q^* = - k \mathrm{T}'_{\mathrm{S}} \mathrm{C} \, \delta \mathrm{S}.$$

In general also these waves are not exceptional, except if the eq. (29_1) reduces to an equality. The associated rays direction is

$$\mathrm{R}^\alpha = (\chi r \mathrm{T} - kf \mathrm{T}'_{\mathrm{S}}) \, \mathrm{U} u^\alpha + kf \mathrm{T}'_{\mathrm{S}} \, \varphi^\alpha$$

and, as is easily verified, it is tangential to the wave surfaces.

Concluding we observe that in the general case: $(c^2 rf)'_{\mathrm{S}} \neq 0$, it is possible to ensure the spatial orientation of all the waves under suitable thermodynamical conditions, but obviously the speed are all changed and are not symmetric with respect to the zero. A set of possible thermodynamical inequalities is given by

$$cfr'_p \geq 1 \quad , \quad \chi r \mathrm{T} \geq kf \mathrm{T}'_{\mathrm{S}} \quad , \quad \mathrm{T}'_{\mathrm{S}} \geq 0 \quad , \quad (c^2 rf)'_{\mathrm{S}} \leq 0 \quad , \quad c^2 rf \mathrm{T}'_p - \mathrm{T} \geq 0 ,$$
$$c^2 r \, (c^2 fr'_p - 1) \, (\chi r \mathrm{T} - kf \mathrm{T}'_{\mathrm{S}}) \geq k \, (\mathrm{T} - c^2 rf \mathrm{T}'_p) \, (c^2 rf)'_{\mathrm{S}}$$

to which one of the following must be associated:

$$q_n \frac{\mathrm{V}_h}{c} < \frac{k \, (c^2 rf \mathrm{T}'_p - \mathrm{T})}{\chi \, (c^2 fr'_p - 1)} \quad , \quad q_n \frac{\mathrm{V}_t}{c} < \frac{k \, (c^2 rf \mathrm{T}'_p - \mathrm{T})}{\chi \, (\chi r \mathrm{T} - kf \mathrm{T}'_{\mathrm{S}})}$$

where V_h and V_t are given by (27) and (30) respectively, and $q_n = \mathrm{Q}/\sqrt{-\mathrm{C}}$.

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