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Nonlocal effects on the stress distribution in an elastic half-space

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Meccanica dei continui. — *Nonlocal effects on the stress distribution in an elastic half-space.* Nota di STAN CHIRITA, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — In questa Nota si dà una rappresentazione di tipo Galerkin nella teoria di elasticità nonlocale e si studia il problema del semispazio.

1. In this paper we examine the stress distribution in the problem of the nonlocal elastic half-space under arbitrary distributed surface tractions. As a general rule, the components of the stress tensor are affected by nonlocal effects. It is interesting to note that there is a case in which the stress distribution is the same as in the classical elasticity. In all cases, the displacement distribution is modified in nonlocal elasticity.

2. We recall the fundamental equations governing the linear theory of homogeneous and isotropic nonlocal elastic solids. In this connection we confine our attention to the equilibrium case, refer to rectangular cartesian coordinates (x_1, x_2, x_3) and we use the notation $x = (x_1, x_2, x_3)$. In what follows, unless otherwise specified, Latin subscripts are understood to range over the integers (1, 2, 3), whereas Greek subscripts are confined to the range (1, 2), and subscripts preceded by a comma denote differentiation with respect to the corresponding cartesian coordinate.

Let V be the region occupied by an elastic solid, and let ∂V the boundary surface of V. The basic equations of linear theory of nonlocal elasticity are [2], [4]

– the equilibrium equations

$$t_{ji,j} + f_i = 0 \,, \tag{1}$$

– the constitutive equations

$$t_{ij} = \lambda e_{rr} \delta_{ij} + 2 \mu e_{ij} + \int_V [\lambda' (x - y) e_{rr} (y) \delta_{ij} + 2 \mu' (x - y) e_{ij} (y)] \, dy, \tag{2}$$

– the geometrical equations

$$2 e_{ij} = u_{i,j} + u_{j,i} \,. \tag{3}$$

In these equations we used the following notations: t_{ij}-the components of the stress tensor, u_i-the components of the displacement vector, f_i-the components of the body force vector, e_{ij}-the components of the strain tensor,

(*) Nella seduta del 10 giugno 1976.

λ, μ-the local elastic constants of the material, λ', μ'-the nonlocal characteristics of the material. To the equations (1)-(3) we adjoin the boundary condition

$$t_{ji}\, n_j = \tilde{t}_i\,, \qquad \text{on} \quad \partial V, \tag{4}$$

where n_j are components of the outward unit normal to ∂V and $\tilde{t}_i$ are prescribed functions.

Substituting (2), (3) in (1) we obtain the displacement equations of equilibrium in the linear theory of nonlocal elasticity

$$\begin{aligned} &\mu \Delta u_i + (\lambda + \mu)\, u_{r,ri} + \\ &+ \frac{\partial}{\partial x_j} \int_V [\lambda'(x-y)\, u_{r,r}(y)\, \delta_{ij} + \mu'(x-y)\,(u_{i,j}(y) + u_{j,i}(y))]\, dy + f_i = 0\,. \end{aligned} \tag{5}$$

Let V be the entire space E_3. According to the axiom of attenuating neighborhoods, the nonlocal moduli λ', μ' must die out fast with $|x_i - y_i| \to \infty$. Therefore, if we assume that [2], [5]

$$\lambda'(|x_i - y_i|) = \lambda_1\, G(|x_i - y_i|)\;, \quad \mu'(|x_i - y_i|) = \mu_1\, G(|x_i - y_i|)\;, \tag{6}$$

$$\lambda_1\,, \mu_1 = \text{const.},$$

then

$$G(|x_i - y_i|) \xrightarrow[|x_i - y_i| \to \infty]{} 0\,. \tag{7}$$

Moreover, we suppose that $u_i(x)$ and $G(|x|)$ are sufficiently smooth functions. Accordingly, we may interchange the derivative and the integration on the displacement vector in (5). In view of this fact and using the notations

$$\begin{aligned} \Lambda\varphi &\equiv (\lambda + 2\,\mu)\,\varphi + (\lambda_1 + 2\,\mu_1) \int_{E_3} G(|x - y|)\, \varphi(y)\, dy, \\ L\varphi &\equiv \mu\varphi + \mu_1 \int_{E_3} G(|x - y|)\, \varphi(y)\, dy, \end{aligned} \tag{8}$$

we derive for the displacement the following representation of the Galerkin type

$$u_i = \Delta\Lambda\Phi_i - \frac{\partial^2}{\partial x_i\, \partial x_r}(\Delta - L)\, \Phi_r\,, \tag{9}$$

where Φ_r satisfies the equation

$$\Delta^2\, \Lambda L \Phi_r = -f_r\,. \tag{10}$$

As a particular case, when $\lambda_1 = \lambda$, $\mu_1 = \mu$ we obtain the classical representation

$$u_i = (\lambda + 2\mu)\,\Delta\Phi_i - (\lambda + \mu)\,\frac{\partial^2}{\partial x_i\,\partial x_r}\,\Phi_r\,, \tag{11}$$

where in this case Φ_r satisfies the equation

$$\mu(\lambda + 2\mu)\,\Delta^2 L'\,\Phi_r = -f_r \quad ; \quad L'\varphi \equiv \varphi + \int_{E_3} G(|x - y|)\,\varphi(y)\,dy. \tag{12}$$

It is not difficult to verify that the above representation indeed satisfies the displacement equation (5).

3. Let V from now on be the open half-space ($-\infty < x_2 , x_3 < \infty$; $0 < x_1 < \infty$) so that ∂V is the plane $x_1 = 0$. The boundary conditions (4) take the form

$$t_{\alpha 1}(0 , x_2 , x_3) = q_\alpha(x_2 , x_3) \quad , \quad t_{31}(0 , x_2 , x_3) = 0, \tag{13}$$

where $q_\alpha(x_2 , x_3)$ are prescribed functions on $x_1 = 0$, so that

$$\int_{E_2} |q_\alpha(x_2 , x_3)|\,dx_2\,dx_3 < \infty.$$

We suppose that the half-space is occupied by an elastic medium with nonlocal interactions. The nonlocal moduli λ' and μ', are expected to change very sharply as we move from the surface, $x_1 = 0$ to within the half-space. According to the axiom of attenuating neighborhoods, they must die out fast with $|x_i - y_i| \to \infty$. Thus, we assume [3]

$$G(|x - y|) = \mathcal{G}(|x' - y'|)\,\delta(x_1 - y_1) \quad , \quad x' \equiv (x_2 , x_3) \quad , \quad y' \equiv (y_2 , y_3). \tag{14}$$

We suppose that the body forces are absent. In order to solve the problem of the half-space we assume $\Phi_\alpha = \varphi_\alpha$, $\Phi_3 = 0$, so that the representation (9) becomes

$$u_\alpha = \Delta\Lambda\varphi_\alpha - \frac{\partial}{\partial x_\alpha}(\Lambda - L)\,\varphi_{\rho,\rho} \quad , \quad u_3 = -\frac{\partial}{\partial x_3}(\Lambda - L)\,\varphi_{\rho,\rho}\,, \tag{15}$$

$$\Delta^2\,\Lambda L\varphi_\alpha = 0\,, \tag{16}$$

where we have used the notation

$$\Lambda\psi(x_1 , x') \equiv (\lambda + 2\mu)\,\psi(x_1 , x') + (\lambda_1 + 2\mu_1)\int_{E_2} \mathcal{G}(|x' - y'|)\,\psi(x_1 , y')\,dy', \tag{17}$$

$$L\psi(x_1 , x') \equiv \mu\psi(x_1 , x') + \mu_1 \int_{E_2} \mathcal{G}(|x' - y'|)\,\psi(x_1 , y')\,dy'\,.$$

Corresponding to the representation (15) we have

$$
(18)\qquad
\begin{aligned}
t_{\alpha\beta} &= \Delta L (\Lambda - 2\, L)\, \varphi_{\rho,\rho}\, \delta_{\alpha\beta} + \Delta \Lambda L\, (\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}) - \\
&\quad - 2\, \frac{\partial^2}{\partial x_\alpha\, \partial x_\beta}\, L\, (\Lambda - L)\, \varphi_{\rho,\rho}\,, \\
t_{\alpha 3} &= \Delta \Lambda L \varphi_{\alpha,3} - 2\, \frac{\partial^2}{\partial x_\alpha\, \partial x_3}\, L\, (\Lambda - L)\, \varphi_{\rho,\rho}\quad , \\
t_{33} &= \Delta L (\Lambda - 2\, L)\, \varphi_{\rho,\rho} - 2\, \frac{\partial^2}{\partial x_3^2}\, L\, (\Lambda - L)\, \varphi_{\rho,\rho}\,.
\end{aligned}
$$

If we apply the Fourier transform with respect to x' to equation (16), considering x_1 as a parameter, we obtain

$$
(19)\qquad \left(\frac{d^2}{dx_1^2} - \xi^2\right)^2 \mathscr{F}(\varphi_\alpha) = 0 \quad , \quad \xi^2 = \xi_2^2 + \xi_3^2\,,
$$

where $\mathscr{F}(\varphi_\alpha)$ is the Fourier transform of the function φ_α with respect to x'. If we use the regularity condition for $x_1 \to \infty$, it is easy to see that the general solution of (19) is

$$
(20)\qquad \mathscr{F}(\varphi_\alpha) = (A_\alpha + B_\alpha\, x_1)\, e^{-\xi x_1},
$$

where A_α and B_α are unknown constants.

Taking into account the relations [6]

$$
(21)\qquad
\begin{aligned}
\mathscr{F}(\varphi_{\alpha,1}) &= [B_\alpha - \xi\, (A_\alpha + B_\alpha\, x_1)]\, e^{-\xi x_1}\ ; \\
\mathscr{F}(\varphi_{\alpha,11}) &= [-2\, \xi\, B_\alpha + \xi^2\, (A_\alpha + B_\alpha\, x_1)]\, e^{-\xi x_1}\ ; \\
\mathscr{F}(\varphi_{\alpha,111}) &= [3\, \xi^2\, B_\alpha - \xi^3\, (A_\alpha + B_\alpha\, x_1)]\, e^{-\xi x_1}\ ,
\end{aligned}
$$

from the boundary conditions (13) and the relations (18), (20) we deduce

$$
(22)\qquad
\begin{aligned}
&B_1 = \frac{1}{(\bar\lambda + \bar\mu)\, \xi^2} \left[\frac{i\xi_2\, \mathscr{F}(q_2)}{2\, (\bar\lambda + 2\bar\mu)\, \xi} + \frac{\mathscr{F}(q_1)}{2\bar\mu}\right] \quad , \quad B_2 = \frac{\mathscr{F}(q_2)}{2\bar\mu\, (\bar\lambda + 2\bar\mu)\, \xi^2}\,, \\
&\bar\lambda B_1 = (\bar\lambda + \bar\mu)\, \xi A_1 \quad , \quad B_2 = \xi A_2 \quad , \quad \bar\lambda = \lambda + \lambda_1\, \mathscr{F}(\mathscr{G})\, (\xi)\,, \\
&\bar\mu = \mu + \mu_1\, \mathscr{F}(\mathscr{G})\, (\xi)\,.
\end{aligned}
$$

Then, we obtain

$$
(23)\qquad
\begin{aligned}
\mathscr{F}(t_{11}) &= \{\mathscr{F}(q_1) + [\xi \mathscr{F}(q_1) + i\xi_2\, \mathscr{F}(q_2)]\, x_1\}\, e^{-\xi x_1}\,, \\
\mathscr{F}(t_{1j}) &= \left\{\mathscr{F}(q_2)\, \delta_{j2} + \frac{i\xi_j}{\xi}\, [\xi \mathscr{F}(q_1) + i\xi_2\, \mathscr{F}(q_2)]\, x_1\right\} e^{-\xi x_1}, \qquad j = 2\,, 3.
\end{aligned}
$$

Integral representations, in complex form, for the components of stress and displacement now follow from (15), (18), (20), (23) with the aid of the inversion of the Fourier transform.

As a final application we consider here the problem of a half-space subjected to a normal concentrated traction, applied in the origin of the coordinate system. Thus,

$$\mathscr{F}(q_1) = 1 \quad , \quad \mathscr{F}(q_2) = 0 \quad , \quad A_2 = B_2 = 0 \quad , \quad \varphi_2 = 0. \tag{24}$$

With these special assumptions, from (18), (22) we deduce

$$\begin{aligned}
&\mathscr{F}(\varphi_1) = \frac{1}{2\bar{\mu}(\bar{\lambda}+\bar{\mu})}\left[\frac{\bar{\lambda}}{(\bar{\lambda}+\bar{\mu})\xi^3} + \frac{x_1}{\xi^2}\right] e^{-\xi x_1} , \\
&\mathscr{F}(t_{11}) = (1 + \xi x_1)\, e^{-\xi x_1} , \\
&\mathscr{F}(t_{1j}) = i\xi_j x_1 e^{-\xi x_1} , \\
&\mathscr{F}(t_{jk}) = \left\{\frac{\bar{\lambda}}{\bar{\lambda}+\bar{\mu}}\,\delta_{jk} + \left[\frac{\bar{\mu}}{(\bar{\lambda}+\bar{\mu})\xi^2} - \frac{x_1}{\xi}\right]\xi_j \xi_k\right\} e^{-\xi x_1} , \; j,k=2,3 .
\end{aligned} \tag{25}$$

According to the relations [6]

$$\begin{aligned}
&\frac{1}{2\pi}\mathscr{F}\left(\frac{1}{R}\right) = \frac{1}{\xi} e^{-\xi x_1} , \quad (x_1 > 0) \; ; \\
&-\frac{1}{2\pi}\mathscr{F}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right)\right] = e^{-\xi x_1} \; ; \\
&\frac{1}{2\pi}\mathscr{F}\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{R}\right)\right] = \xi e^{-\xi x_1} \; ; \\
&-\frac{1}{2\pi}\mathscr{F}[\ln(R + x_1)] = \frac{1}{\xi^2} e^{-\xi x_1} \; ; \\
&R^2 = x_1^2 + x_2^2 + x_3^2 ,
\end{aligned} \tag{26}$$

from $(25)_{2,3}$ we obtain

$$\begin{aligned}
&t_{11} = -\frac{1}{2\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right) + \frac{x_1}{2\pi}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{R}\right) , \\
&t_{1j} = \frac{1}{2\pi}\frac{\partial}{\partial x_j}\left[x_1 \frac{\partial}{\partial x_1}\left(\frac{1}{R}\right)\right] , \qquad j = 2, 3 ,
\end{aligned} \tag{27}$$

as in the classical theory. As is apparent from $(25)_4$, since $\bar{\lambda} = \bar{\lambda}(\xi)$, $\bar{\mu} = \bar{\mu}(\xi)$, the nonlocal effects are present in the components of the stress tensor. It should be noted that in the particular case $\lambda_1 = \lambda$, $\mu_1 = \mu$ the stresses t_{jk} $(j, k = 2, 3)$ in $(25)_4$ coincide with the corresponding well-known results in ordinary elasticity theory. The integrals to which one is led in this case are elementary and may be explicitly evaluated again by making use of (26). However, the nonlocal effects are present in the displacement expressions

$$\mathscr{F}(u_1) = -\frac{1}{2\,\bar{\mu}\xi}\left(\frac{\bar{\lambda}+2\,\bar{\mu}}{\bar{\lambda}+\bar{\mu}}+\xi x_1\right)e^{-\xi x_1}\,,$$

(28)

$$\mathscr{F}(u_j) = \frac{i\xi_j}{2\,\bar{\mu}\xi^2}\left[\frac{\bar{\mu}}{\bar{\lambda}+\bar{\mu}}-\xi x_1\right]e^{-\xi x_1}\,, \qquad j = 2\,,3.$$

For a given elastic material, we may determine $t_{jk}\,(j\,,k = 2\,,3)$ and u_i by inverting the Fourier transform in $(25)_4$, (28).

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