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Existence and stability of solutions for autonomous multivalued differential equations in Banach space

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Equazioni differenziali ordinarie. — *Existence and stability of solutions for autonomous multivalued differential equations in Banach space.* Nota di FRANCESCO S. DE BLASI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema di esistenza per il problema di Cauchy $\dot{x} \in F(x)$, $x(0) = x_0$ in uno spazio di Banach riflessivo. Si suppone F a valori compatti convessi, semicontinua superiormente e γ-Lipschitziana (γ è la misura di non compattezza di Hausdorff). Il teorema ottenuto estende un risultato analogo recentemente enunciato da Muhsinov [12] nel caso di uno spazio di Hilbert separabile. Inoltre, impiegando la nozione di differenziale multivoco introdotta in [7], si dimostra per lo stesso problema un teorema di stabilità.

1. The aim of this Note is to prove two results on the existence and stability of solutions for autonomous multivalued differential equations

$$\dot{x} \in F(x) \qquad x(0) = x_0 \tag{1.1}$$

in a Banach space. Here F is a map from a reflexive Banach space Y to the space $K_0(Y)$ consisting of all non void compact convex subsets of Y, and $\dot{x}$ denotes the strong derivative of x.

In Section 2 an existence theorem for (1.1) is given. F is supposed to be *compact convex valued, upper semicontinuous* (= u.s.c.) and γ-*Lipschitz* (γ is the Hausdorff measure of noncompactness [14], [7]). For the proof we use, in a modified and simplified version, certain results from [2], [3], [6], [15]. Further recent contributions to the theory of multivalued differential equations in infinite dimensional spaces can be found in [5], [12]. (For the single valued case see [1], [4], [11], [15]).

In Section 3 we use the notion of multivalued differential D of F (see [7]) to obtain a theorem on the asymptotic stability of the origin for (1.1), by the first approximation method. This is accomplished supposing that, for the first approximation of (1.1) $\dot{x} \in D(x)$, there exists a Lyapunov functional satisfying some natural conditions.

In conclusion we observe that some of the results of the paper (e.g. Theorem 1) can be formulated also for nonautonomous multivalued differential equations $\dot{x} \in F(t, x)$.

2. In this section an existence theorem for (1.1) is proved. In the sequel Y denotes a real reflexive Banach space, Y^* its dual. We follow the terminology of [10].

(*) Nella seduta del 10 giugno 1976.

LEMMA 1 ([10], p. 88). *If* $x : [0, T] \to Y$ *is* (a_1) *of strong bounded variation,* (a_2) *almost everywhere (= a.e.) weakly differentiable with derivative* y, *then* y *is Bochner integrable. If* x *is also* (a_3) *weakly absolutely continuous then it can be expressed as the indefinite integral of* y,

$$x(t) = x_0 + \int_0^t y(s)\, ds, \qquad t \in [0, T].$$

The following lemma is known. Nevertheless the proof is included to make the paper self contained.

LEMMA 2. *Let the sequence* $\{x_n\}$,

$$x_n(t) = x_0 + \int_0^t \dot{x}_n(s)\, ds, \qquad t \in [0, T],$$

($\dot{x}_n$ *strongly measurable) converge uniformly to the continuous function* x. *Let* $\|\dot{x}_n(t)\| \leq M$ *a.e. in* $[0, T]$. *Then for almost all* $t \in [0, T]$, x *possesses a strong derivative* $\dot{x}$ *which is Bochner integrable and*

$$x(t) = x_0 + \int_0^t \dot{x}(s)\, ds, \qquad t \in [0, T]. \tag{2.1}$$

Proof. The function x satisfies the hypotheses $a_1 - a_3$ of Lemma 1; (a_1) is obvious, (a_2) follows from a result of [13]. To check (a_3) let $x^* \in Y^*$ and consider the sequence $\{g_n\}$, $g_n(s) = \langle x^*, \dot{x}_n(s)\rangle$ $s \in [0, T]$. By Pettis' theorem ([10], p. 72) the functions g_n are measurable. Furthermore $\{g_n\}$ is bounded in $L_1[0, T]$ since a.e. we have $|g_n(s)| \leq \|x^*\| \, \|\dot{x}_n(s)\| \leq \|x^*\| M$. Moreover the countable additivity of the integrals $\int_E g_n(s)\, ds$, E any measurable subset of $[0, T]$, is uniform with respect to n. By a known result ([9], p. 292) $\{g_n\}$ is weakly sequentially compact in $L_1[0, T]$ and there exist $g \in L_1[0, T]$ and subsequence $\{g_{k(n)}\}$ which converges weakly to g. This implies ([9], p. 291)

$$\lim_{n\to\infty} \int_0^t g_{k(n)}(s)\, ds = \int_0^t g(s)\, ds, \qquad t \in [0, T].$$

Thus

$$\langle x^*, x(t)\rangle = \langle x^*, x_0 + \lim_{n\to\infty} \int_0^t \dot{x}_{k(n)}(s)\, ds\rangle$$

yields

$$\langle x^*, x(t)\rangle = \langle x^*, x_0\rangle + \int_0^t g(s)\, ds, \qquad t \in [0, T]$$

and (a_3) is satisfied.

A function $x : [0, T] \to Y, T > 0$, defined by $x(t) = x_0 + \int_0^t y(s)\, ds$, $t \in [0, T]$ (the integral in the sense of Bochner) and such that $\dot{x}(t) \in F(x(t))$ a.e. in $[0, T]$ is called *solution* of (1.1).

The next lemma provides a sufficient condition in order for (1.1) have a solution. Denote by $K_0(Y)$ the space of all non void compact convex subsets of Y with Hausdorff metric d and let $U \subset Y$ be a non void and open set. Among the hypotheses (i), (ii), (iii) on $F : U \to K_0(Y)$ probably (iii) looks as the less natural. However, as it will be seen later, (iii) is certainly satisfied if F is supposed to be γ-Lipschitz or, in particular, completely continuous. Set $S = \{x \in Y : \|x\| < 1\}$.

Recall that $F : U \to K_0(Y)$ is said to be *upper semicontinuous* (in U) if for every $x \in U$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x + h) \subset F(x) + \varepsilon S$ if $\|h\| < \delta$. F is said to be γ-*Lipschitz*, with constant $k \geq 0$, if for every non void bounded subset $A \subset U$ we have $\gamma(F(A)) \leq k\gamma(A)$.

The following lemma is a variant of some known results [2], [6]. Since the proof which we present seems to be new and quite elementary, it is included. For the integral of a multifunction we refer to [8]. For $P \in K_0(Y)$, $\|P\|$ stands for $d(P, 0)$.

LEMMA 3. *Let* Y *be a reflexive Banach space. Suppose*: (i) $F : U \to K_0(Y)$, $U = \{x \in Y : \|x - x_0\| \leq r\}$ $(r > 0)$, *is upper semicontinuous*, (ii) $\|F(x)\| \leq M$, $(M > 0)$ $x \in U$ *and*, (iii) *there exist* $Q \in K_0(Y)$ *and* $T > 0$ *such that* $TM < r$ *and* $x_0 + \bigcup_{t \in [0,T]} t \operatorname{co} F(Q) \subset Q$. *Then* (1.1) *has at least one solution defined on* $[0, T]$.

Proof. For any fixed integer $n \geq 2$ consider the partition of $[0, T]$ by means of the points $t_i = \frac{i}{n} T$, $i = 0, 1, \cdots, n$. Set $I_i = [t_i, t_{i+1})$, $0 \leq i \leq n - 2$, $I_{n-1} = [t_{n-1}, t_n]$ and denote by χ_{I_i} the characteristic function of I_i. Define $x_n : [0, T] \to Y$ by

$$x_n(t_0) = x_0, \; x_n(t) = x_n(t_i) + \int_{t_i}^{t} f_i^n\, ds, \; t \in [t_i, t_{i+1}]$$

$(i = 0, 1, \cdots, n - 1)$ where f_i^n is a point in $F(x_n(t_i))$. This definition is meaningful since any point $x_n(t_i)$ is in U, being $\|x_n(t_i) - x_0\| \leq \frac{i}{n} TM < r$. Thus if $t \in [0, T]$, say $t \in I_i$, we have

$$x_n(t) = x_0 + \sum_{h=0}^{i-1} \int_{t_h}^{t_{h+1}} f_h^n\, ds + \int_{t_i}^{t} f_i^n\, ds$$

$$= x_0 + \int_0^t \left[\sum_{h=0}^{i} f_h^n \chi_{I_h}(s)\right] ds = x_0 + \int_0^t \dot{x}_n(s)\, ds.$$

The sequence $\{x_n\}$ is equicontinuous. Furthermore, for every $t \in [0, T]$, $x_n(t) \in Q$. This is trivial for $t \in [t_0, t_1]$. Suppose $x_n(t) \in Q$, $t \in [t_0, t_h]$. Then, for $t \in [t_h, t_{h+1}]$, we have

$$x_n(t) = x_0 + t \left[\sum_{i=0}^{h-1} \frac{t_{i+1} - t_i}{t} f_i^n + \frac{t - t_h}{t} f_h^n \right]$$

$$\in x_0 + t \left[\sum_{i=0}^{h-1} \frac{t_{i+1} - t_i}{t} F(x_n(t_i)) + \frac{t - t_h}{t} F(x_n(t_h)) \right] \subset Q.$$

By the Ascoli–Arzelà theorem there exists a subsequence, which we denote again by $\{x_n\}$, which converges uniformly to a continuous function $x : [0, T] \to Y$. By Lemma 2, $\dot{x}$ exists a.e. and (2.1) holds.

We claim that x is a solution of (1.1). To this end define

$$z_n : [0, T] \to Y, \; z_n(t) = \sum_{i=0}^{n-1} x_n(t_i) \chi_{I_i}(t).$$

Clearly $z_n \to x$ uniformly. Let $t \in (0, T)$ be a point such that $\dot{x}(t)$ exists. Let $\varepsilon > 0$. Suppose $h > 0$ and sufficiently small (if $h < 0$ the argument is similar). We have

$$x_n(t+h) - x_n(t) = \int_t^{t+h} \dot{x}_n(s)\, ds \in \int_t^{t+h} F(z_n(s))\, ds.$$

Since F is uniformly u.s.c. in the compact set Q and $z_n \to x$ uniformly in $[0, T]$, $(x_n(t), x(t) \in Q)$ there exists an integer $k \geq 1$ such that $F(z_n(s)) \subset F(x(s)) + \varepsilon S$, if $n \geq k$, for all $s \in [0, T]$. Thus

$$x_n(t+h) - x_n(t) \in \int_t^{t+h} F(x(s))\, ds + h\varepsilon S \qquad n \geq k$$

from which, letting $n \to \infty$, hence dividing by h, we obtain

$$[x(t+h) - x(t)]/h \in \frac{1}{h} \int_t^{t+h} F(x(s))\, ds + \varepsilon S.$$

Since $s \to F(x(s))$ is u.s.c. at $s = t$ there exists $h_0 > 0$ such that for every $s \in [t, t+h]$, $0 < h < h_0$, we have $F(x(s)) \subset F(x(t)) + \varepsilon S$. Therefore

$$[x(t+h) - x(t)]/h \in \frac{1}{h} \int_t^{t+h} F(x(t))\, ds + 2\varepsilon S = F(x(t)) + 2\varepsilon S,$$

$F(x(t))$ being compact and convex. Letting $h \to 0$ we obtain $\dot{x}(t) \in F(x(t)) + 2\varepsilon S$, which completes the proof.

THEOREM 1. *Let* Y *be a reflexive Banach space. Let hypotheses* (i), (ii) *of Lemma* 3 *be satisfied. Suppose* F *is* γ*-Lipschitz with constant* $k \geq 0$. *Let* $T > 0$ *be such that* $kT < 1$ *and* $TM < r$. *Then* (1.1) *has at least one solution defined on* $[0, T]$.

Proof. It suffices to apply Lemma 3 since, under the stated hypotheses, the γ-Lipschitz function F satisfies hypothesis (iii) of Lemma 3, (see [15]).

In [12] Muhsinov states, without proof, a similar theorem in a separable Hilbert space.

COROLLARY 1. *Let* Y *be a reflexive Banach space. Suppose*: (i) $F : Y \to K_0(Y)$ *is upper semicontinuous*, (ii) $\| F(x) \| \leq M$, $x \in Y$ *and*, (iii) F *is* γ*-Lipschitz with constant* $k \geq 0$. *Then* (1.1) *has at least one solution* $x : [0, \infty] \to Y$ *defined all over* $[0, \infty)$.

Proof. Let $T > 0$ be such that $kT < 1$. Fix $r > MT$. By Theorem 1, (1.1) has at least one solution x which is defined on $[0, T]$. To continue this solution on $[T, 2T]$, $[2T, 3T]$, $\cdots$ one has to take (1.1), replace x_0 by $x(T)$, $x(2T)$, $\cdots$ and apply Theorem 1.

3. The aim of this section is to prove a criterion for the asymptotic stability of the zero solution of (1.1) using the direct method of Lyapunov.

Denote by B(Y) (resp. B(Y^*)) the set of all non void bounded subsets of Y (resp. Y^*). For $A \in B(Y)$, $\varphi \in B(Y^*)$ define

$$\langle \varphi, A \rangle^+ = \sup \{ \langle f, a \rangle : f \in \varphi, a \in A \}$$

$$\langle \varphi, A \rangle^- = \inf \{ \langle f, a \rangle : f \in \varphi, a \in A \}.$$

If φ, A are singleton $\langle \varphi, A \rangle^+ = \langle \varphi, A \rangle^- = \langle \varphi, A \rangle$.

A nonnegative function $V : Y \to [0, \infty)$ is called a *Lyapunov functional* if there exists $r > 0$ such that V and grad V are continuous in $S_r = \{x \in Y : \| x \| < r\}$ and, there exist constants $\alpha, \beta > 0$ such that $\alpha \| x \|^2 \leq V(x) \leq \beta \| x \|^2$, $x \in S_r$.

Denote by d the Hausdorff distance in the space $C_0(Y)$ consisting of all non void bounded closed and convex subsets of Y.

Let U be a (non void) open subset of Y. $F : U \to K_0(Y)$ is said to be *differentiable* [7] at $x \in U$ if there exists a map $D_x : Y \to C_0(Y)$ such that: (a) D_x is u.s.c., homogeneous i.e. $D_x(ty) = tD_x(y)$, $t \geq 0$ $y \in Y$ and, (b) there exists $\delta > 0$ such that $d(F(x+h), F(x) + D_x(h)) = o(h)$, when $\| h \| < \delta$, and $\lim_{h \to 0} o(h)/\| h \| = 0$.

THEOREM 2. *Let* (1.1) *be given. Let* Y *be a reflexive Banach space. Let* $F : Y \to K_0(Y)$, $F(0) = 0$, *satisfy the hypotheses of Corollary 1 and, moreover, suppose that* F *is differentiable at the origin with multivalued differential* D.

Let there exist a Lyapunov functional V *such that*

$$(3.1)\qquad \langle \text{grad}\, V(x)\,,\, D(x)\rangle^{+} \leq -aV(x)\,,\ a > 0 \qquad x \in S_r$$

$$(3.2)\qquad \|\text{grad}\ V(x)\| \leq b\|x\|\,,\ b > 0 \qquad x \in S_r\,.$$

Then the origin is asymptotically stable for (1.1).

Proof. Since F is differentiable at the origin and F (0) = 0, there exist a bounded closed convex valued map $D : Y \to C_0(Y)$ and a constant $0 < \delta < r$ such that

$$(3.3)\qquad d(F(x)\,,\,D(x)) = o(x)\,, \qquad \text{if}\ \ \|x\| < \delta\,,$$

and $\lim_{x\to 0} o(x)/\|x\| = 0$. Choose $\delta > 0$ satisfying $-a + b\,\delta/\alpha \leq -a/2$. Set $o^1(x) = o(x) + \|x\|^2$. Since $\lim_{x\to 0} o^1(x)/\|x\| = 0$ there exists $0 < \delta_1 < \delta$ such that both (3.3) and

$$(3.4)\qquad -a + \frac{b}{\alpha}\,\frac{o^1(x)}{\|x\|} \leq -a + \frac{b}{\alpha}\,\delta \leq -\frac{a}{2}$$

hold for all $\|x\| < \delta_1$. From (3.3) and the definition of Hausdorff distance d we have $F(x) \subset D(x) + o^1(x)\,S$ if $\|x\| < \delta_1$. Let $H > 1$ be such that $(\beta/\alpha)^{1/2} < \delta_1/K < \delta_1/2$.

Denote by x any solution of (1.1), defined for $t \geq 0$, with initial value x_0 satisfying $\|x_0\| < \delta_1/K$. By Corollary 1 there exists at least one such solution. As long as $x(t)$ satisfies $\|x(t)\| < \delta_1$ we have a.e.

$$\frac{d}{dt}V(x(t)) = \langle \text{grad}\, V(x(t))\,,\,\dot{x}(t)\rangle$$
$$\leq \langle \text{grad}\, V(x(t))\,,\,D(x(t)) + o^1(x(t))\,S\rangle^{+}$$
$$\leq -aV(x(t)) + o^1(x(t))\,b\,\|x(t)\|.$$

Thus, if $x(t) \neq 0$,

$$\frac{d}{dt}V(x(t)) \leq -aV(x(t)) + \frac{o^1(x(t))}{\|x(t)\|}\,b\,\|x(t)\|^2,$$

from which, using (3.4), we obtain

$$(3.5)\qquad \frac{d}{dt}V(x(t)) \leq -\frac{a}{2}V(x(t))\,,$$

a.e. and for all $t \geq 0$ such that $x(t) \neq 0$ and $\|x(t)\| < \delta_1$.

This inequality is trivially satisfied for all t such that $x(t) = 0$ since, in this case, both members vanish. Furthermore it is easy to see that

$\|x_0\| < \delta_1/K$ implies $\|x(t)\| < \delta_1, t \geq 0$. Thus (3.5) yields $V(x(t)) \leq \leq V(x_0) \exp\left(-\frac{a}{2}t\right)$ and $\|x(t)\| \leq \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} \|x_0\| \exp\left(-\frac{a}{4}t\right)$, $t \geq 0$, which completes the proof.

Example. Let $\overline{S}(a, r) = \{x \in R^n : \|x - a\| \leq r\}$. Consider the equation in a neighborhood of the origin of R^n

$$\dot{x} \in F(x), \quad F(x) = -x + \overline{S}\left(0, \frac{1}{2}\|x\| + r(x)\right)$$

where $r(x) = 0$ if $x = 0$, $r(x) = \frac{\|x\|^2}{2} \sin \frac{1}{\|x\|}$ if $x \neq 0$. F has at the origin the multivalued differential $D(x) = -x + \overline{S}\left(0, \frac{1}{2}\|x\|\right)$. Furthermore $V(x) = \langle x, x \rangle, x \in R^n$, is a Lyapunov functional satisfying (3.2) and also (3.1) for

$$\langle \operatorname{grad} V(x), D(x) \rangle^+ = \sup 2\left\{\langle x, -x + \frac{1}{2}t\|x\| \rangle : 0 \leq \|t\| \leq 1\right\} \leq -\|x\|^2.$$

Then Theorem 2 applies, i.e. the origin is asymptotically stable for the given equation.

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